In this section we will learn formulas for trig functions that involve the sum or difference of two angles. Let's start with the sum and difference formulas for cosine and sine:

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta
\end{align*}
\]

To help you remember these formulas, let's think of them this way:

To expand a sum/difference within cosine: (multiply cosines together) [opposite sign] (multiply sines together)
*Just remember "CGO": For Cosine, Group like functions and use the Opposite sign*

To expand a sum/difference within sine: (multiply sine by cosine) [same sign] (multiply cosine by sine)
*Remember "SMS": For Sine, Mix the functions and use the Same sign*

*In both cases, the first trig function after the equals sign is the same as the original trig function.*

**YOU NEED TO MEMORIZE THESE FORMULAS!!!**

For example, let's expand \( \cos(45^\circ - 30^\circ) \). We remember that for cosine, we are going to multiply together like functions and put the opposite sign between them. So since the original problem has a minus sign, we'll use a plus sign in our expansion. Thus:

\[
\cos(45^\circ - 30^\circ) = \cos _____^\circ \cos _____^\circ + \sin _____^\circ \sin _____^\circ = _____ \cdot _____ + _____ \cdot _____
\]

Since the original problem was \( \cos(45^\circ - 30^\circ) \), this could have been stated as "Find the exact value of \( \cos(15^\circ) \)."

Let's look at another example where the problem is originally stated in this manner: Find the exact value of \( \sin(120^\circ) \).

For Sines, remember that we Mix sines and cosines, starting with sine of the first angle, and we use the Same sign. We need to decide which two of our commonly-used angles \( (30^\circ, 45^\circ, \text{ and } 60^\circ) \) add or subtract to equal \( 120^\circ \).

\[
\sin(120^\circ) = \sin(____^\circ + ____^\circ) =
\]
Now let's look at a couple examples of angles in radians. A good rule of thumb is that if the denominator is 12, then it is a combination of $\frac{\pi}{4}$ and either $\frac{\pi}{3}$ or $\frac{\pi}{6}$, since the least common denominator of 4 and 3 (and also 4 and 6) is 12.

For help with these problems, remember that: $\frac{\pi}{4} = \frac{12}{12}$, $\frac{\pi}{3} = \frac{12}{12}$, and $\frac{\pi}{6} = \frac{12}{12}$.

Example: Find the exact value of each expression. Remember CGO and SMS!

a) $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\ldots + \ldots\right) = \ldots$

b) $\sin\left(\frac{\pi}{12}\right)$

We can also do these problems in reverse, where we start with an expansion and condense it into a single sine or cosine function. Again, it is helpful to remember CGO and SMS (if like functions are Grouped together, it is an expansion of a Cosine function, and if the functions are Mixed together, then it is an expansion of a Sine function).

Example: Find the exact value of each expression.

a) $\sin 20^\circ \cos 80^\circ - \cos 20^\circ \sin 80^\circ$

Because the sines and cosines are Mixed together, we know this is the expansion of a __________ function.

Completing the final letter of our "SMS" guide, we know that the condensed function will use the ________ sign as the expanded one, so the condensed function will be: ____________________________

Now find the exact value of this trig function.
Example (continued): Find the exact value of each expression.

b) \[ \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} - \sin \frac{5\pi}{12} \sin \frac{7\pi}{12} \]

Because the sines and cosines are grouped together, we know this is the expansion of a ________________ function. Completing the final letter of our "CGO" guide, we know that the condensed function will use the __________________ sign as the expanded one, so the condensed function will be: __________________

Now find the exact value of this trig function.

Now let's learn the sum and difference formulas for tangent:

Notice that the numerator has the same sign as the original function and the denominator has the opposite sign. You do not have to memorize the sum/difference formulas for tangent because they are not used as frequently as the sine and cosine formulas.

Example: Find the exact value of each expression.

a) \[ \tan \left( \frac{5\pi}{12} \right) = \tan \left( \quad \quad + \quad \quad \right) = \]

b) \[ \tan \left( \frac{19\pi}{12} \right) \] This angle is in Quadrant _______. Tangent is __________________ in this quadrant. What is the reference angle? __________ So this is actually the same as ________________________________.

Now rewrite this angle as a sum or difference of two angles and find the exact value.
Example (continued): Find the exact value of each expression.

c) \[
\frac{\tan 40^\circ - \tan 10^\circ}{1 + \tan 40^\circ \tan 10^\circ}
\]

FINDING EXACT VALUES GIVEN THE VALUES OF SOME TRIGONOMETRIC FUNCTIONS

Example: Find the exact value of (a) \(\sin(\alpha + \beta)\), (b) \(\cos(\alpha + \beta)\), (c) \(\sin(\alpha - \beta)\), (d) \(\tan(\alpha - \beta)\) given the following information about \(\alpha\) and \(\beta\): 
\[
\cos \alpha = \frac{\sqrt{5}}{5}, \quad 0 < \alpha < \frac{\pi}{2}, \quad \sin \beta = -\frac{4}{5}, \quad -\frac{\pi}{2} < \beta < 0.
\]

In order to use our sum and difference formulas, we need some more information. For parts (a), (b), and (c), we will need to find \(\sin(\alpha)\) and \(\cos(\beta)\) to plug into our formulas. For part (d) we will need \(\tan(\alpha)\) and \(\tan(\beta)\) also.

The problem tells us that \(\alpha\) is in Quadrant ______. So draw a triangle in that quadrant that has an adjacent side of \(\sqrt{5}\) and a hypotenuse of 5. Now use the Pythagorean Theorem to find the opposite side. Using this completed triangle, fill in the following values: \(\sin(\alpha) = \) _______ and \(\tan(\alpha) = \) _______.

The problem also tells us that \(\beta\) is in Quadrant ______. So draw a triangle in that quadrant that has an opposite side of 4 and a hypotenuse of 5. Now use the Pythagorean Theorem to find the adjacent side side. Using this completed triangle, fill in the following values: \(\cos(\beta) = \) _______ and \(\tan(\beta) = \) _______.

Now we are ready to answer the questions that the original problem asked!
Rewrite all of the values from the previous page so we can have them handy as we work the problem out:

\[ \cos \alpha = \frac{\sqrt{5}}{5}, \sin \alpha = \text{________}, \tan \alpha = \text{________}, \cos \beta = \text{________}, \sin \beta = -\frac{4}{5}, \tan \beta = \text{________} \]

(a) \( \sin(\alpha + \beta) \)

(b) \( \cos(\alpha + \beta) \)

(c) \( \sin(\alpha - \beta) \)

(d) \( \tan(\alpha - \beta) \)
ESTABLISHING IDENTITIES USING THE SUM AND DIFFERENCE FORMULAS

Example: Establish each identity.

a) \( \cos (\pi - \theta) = -\cos \theta \)

b) \( \cos \left( \frac{3\pi}{2} + \theta \right) = \sin \theta \)

c) \( \frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta \)

USING SUM AND DIFFERENCE FORMULAS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Example: Find the exact value of the expression \( \cos \left( \tan^{-1} \frac{4}{3} + \cos^{-1} \frac{12}{13} \right) \).

The outer function is cosine. Remember CGO: \( \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \).

Let \( \alpha = \tan^{-1} \frac{4}{3} \) and \( \beta = \cos^{-1} \frac{12}{13} \). This means that \( \tan \alpha = \frac{4}{3} \) and \( \cos \beta = \frac{12}{13} \). So we already have the value of \( \cos \beta \), but in order to use our sum formula, we still need to find \( \cos \alpha \), \( \sin \alpha \), and \( \sin \beta \). \( \tan \alpha \) and \( \cos \beta \) are both positive, which means both \( \alpha \) and \( \beta \) come from Quadrant _____. For \( \alpha \), we draw a triangle with opposite side of 4 and adjacent side of 3, then use the Pythagorean Theorem to find that the hypotenuse is _______. So using this triangle we see that \( \sin \alpha = ____ \) and \( \cos \alpha = ____ \).

Now we move to \( \beta \), where we draw a triangle with adjacent side of 12 and hypotenuse of 13. Use the Pythagorean Theorem to find that the opposite side is _______. Using this triangle we see that \( \sin \beta = ____ \). Now plug these values into the sum formula for cosine to find the answer.