The product of two vectors, called the **dot product**, results in a scalar answer (not a vector as you might expect). For this reason, the dot product is also called the **scalar product**. The definition of the dot product is shown in the box to the right.

The dot product is commutative, so \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).

The distributive property also applies to the dot product, so \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \).

**Example:** Find the dot product \( \mathbf{v} \cdot \mathbf{w} \) if \( \mathbf{v} = \langle 2, 2 \rangle \) and \( \mathbf{w} = \langle 1, 2 \rangle \).

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**FINDING THE ANGLE BETWEEN TWO VECTORS**

The angle between two vectors can be found by using the formula in the box to the right. If the angle \( \theta \) between two vectors is 0 or \( \pi \), then the vectors are parallel. If the angle between them is \( \frac{\pi}{2} \) (90°), then the vectors are **orthogonal**. "Orthogonal" is another word for **perpendicular**, which means that the vectors meet at a __________ angle. **Two vectors are orthogonal if their dot product is zero.**

**Example:** Find the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \) and state if the vectors are parallel, orthogonal, or neither.

a) \( \mathbf{u} = \mathbf{i} + \mathbf{j} ; \quad \mathbf{v} = -\mathbf{i} + \mathbf{j} \)

First we find the dot product, \( \mathbf{u} \cdot \mathbf{v} : \)

These vectors are ______________________.

b) \( \mathbf{u} = 3\mathbf{i} - 4\mathbf{j} ; \quad \mathbf{v} = 4\mathbf{i} - 3\mathbf{j} \)

First we find the dot product, \( \mathbf{u} \cdot \mathbf{v} : \)

Next we find the magnitudes of \( \mathbf{u} \) and \( \mathbf{v} : \)

\[ \| \mathbf{u} \| = \sqrt{\left( \right)^2 + \left( \right)^2} = \]

\[ \| \mathbf{v} \| = \sqrt{\left( \right)^2 + \left( \right)^2} = \]

Now plug this information into the formula: \( \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \rightarrow \theta = \cos^{-1} \)

These vectors are ______________________
DECOMPOSING A VECTOR INTO TWO ORTHOGONAL VECTORS

Sometimes when given two vectors, \( \vec{v} \) and \( \vec{w} \), it is necessary to decompose \( \vec{v} \) into two parts: one part, \( \vec{v}_1 \), that is parallel to \( \vec{w} \), and another part, \( \vec{v}_2 \), that is orthogonal to \( \vec{w} \). The parallel part, \( \vec{v}_1 \), is called the projection of \( \vec{v} \) onto \( \vec{w} \).

Example: Decompose \( \vec{v} \) into \( \vec{v}_1 \) which is parallel to \( \vec{w} \), and \( \vec{v}_2 \) which is orthogonal to \( \vec{w} \).

a) \( \vec{v} = -3\hat{i} + 2\hat{j}, \quad \vec{w} = 2\hat{i} + \hat{j} \)

First we find \( \vec{v}_1 \). \( \vec{v} \cdot \vec{w} = (-3)(2) + (2)(1) = -6 + 2 = -4 \), and \( \|\vec{w}\| = \sqrt{(2)^2 + (1)^2} = \sqrt{5} \).

So \( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} = \frac{-4}{(\sqrt{5})^2} = -\frac{4}{5} \). Then \( \vec{v}_1 = -\frac{4}{5}(2\hat{i} + \hat{j}) \rightarrow \vec{v}_1 = -\frac{8}{5}\hat{i} - \frac{4}{5}\hat{j} \).

Now \( \vec{v}_2 = \vec{v} - \vec{v}_1 \rightarrow \vec{v}_2 = (-3\hat{i} + 2\hat{j}) - \left(-\frac{8}{5}\hat{i} - \frac{4}{5}\hat{j}\right) \rightarrow \vec{v}_2 = \left(-3 + \frac{8}{5}\right)\hat{i} + \left(2 + \frac{4}{5}\right)\hat{j} \rightarrow \vec{v}_2 = -\frac{7}{5}\hat{i} + \frac{14}{5}\hat{j} \).

b) \( \vec{v} = \hat{i} - 3\hat{j}, \quad \vec{w} = 4\hat{i} - \hat{j} \)