Greek Letters Commonly Used in Trigonometry

- $\alpha = \text{alpha}$
- $\beta = \text{beta}$
- $\theta = \text{theta}$
- $\delta = \text{delta}$
- $\omega = \text{omega}$
- $\phi = \text{phi}$

The angles above represent coterminal angles, which are angles with the same initial and terminal sides (but you may get from the initial side to the terminal side in different ways).

**DEGREES**

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it ends at the starting point (i.e., one complete revolution) is said to be 360 degrees (360°). You probably remember from your high school geometry class that if you go around a circle, you have gone 360°. So, one degree is equal to $\frac{1}{360}$ of a revolution. 90° is a quarter of a revolution (because $90 = \frac{1}{4}$ of 360) and 180° is half a revolution (because $180 = \frac{1}{2}$ of 360).

Obviously, 90° and 180° are quadrantal angles. What are the other 2 quadrantal angle measurements?

_________ and _________
Example 1: Draw each angle and state which quadrant it is in OR if it is a quadrantal angle.

a) $60^\circ$  
b) $-120^\circ$  
c) $540^\circ$  
d) $225^\circ$

**Quick Tip:** A reference angle is the shortest distance from the terminal side of an angle to the x-axis.

**RADIANS**

If you draw an angle whose vertex is in the center of a circle, it is called a central angle. The points where the rays of the angle intersect with the circle create an arc. If the radius of the circle is equal to the length of the arc, then the measure of the angle is 1 radian. A circle whose radius is one is called a unit circle. We will use the unit circle A LOT during this class!

For a circle of radius $r$, a central angle of $\theta$ radians creates (subtends) an arc whose length $s$ is equal to the radius of the circle times the central angle.

\[ s = r \theta \]  

(arc length equals radius times the angle that created the arc)

**The central angle must be in RADIANS in order to use this formula!**

Example 2: Find the missing quantity.

a) $r = 6$ feet, $\theta = 2$ radians  
b) $r = 6$ meters, $s = 8$ meters
CONVERTING BETWEEN DEGREES AND RADIANS

Definition: 1 revolution = 2π radians

And since we already know that 1 revolution = 360°, that means that 2π radians = 360°.

Dividing both sides by 2 gives us: \( \pi \text{ radians} = 180° \). This equation gives us our two conversion formulas:

- If you are given a radian measurement, multiply it by \( \frac{180°}{\pi \text{ rad}} \) to find its degree measurement.
- If you are given a degree measurement, multiply it by \( \frac{\pi \text{ rad}}{180°} \) to find its radian measurement.

To help you remember which fraction to multiply by, just remember that the units of measurement of what you start with have to cancel out and you want to be left with the unit you are trying to convert to.

Example 3: Convert between radians and degrees.

a) \( 225° = \) ______ radians

\[
225° \cdot \frac{\pi \text{ rad}}{180°} \quad \text{Cancel Degrees}
\]

b) \( \frac{11\pi}{6} = \) ______ degrees

\[
\frac{11\pi}{6} \text{ rad} \cdot \frac{180°}{\pi \text{ rad}} \quad \text{Cancel Radians and } \pi \text{ 's}
\]

c) 1 radian \( \approx \) ______ degrees

Example 4: Let’s figure out what the radian measurement is for our quadrantal angles.

a) \( 0° = \) ______ radians 

b) \( 90° = \) ______ radians

c) \( 180° = \) ______ radians 

d) \( 270° = \) ______ radians
Some other commonly used radian angle measurements are $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians. Convert these angles to their degree measurements. What quadrant do these angles fall in?

\[ e) \quad \frac{\pi}{6} \text{ radians} = \underline{\text{______}} \text{ degrees} \quad \text{Quadrant _____} \]

\[ f) \quad \frac{\pi}{4} \text{ radians} = \underline{\text{______}} \text{ degrees} \quad \text{Quadrant _____} \]

\[ g) \quad \frac{\pi}{3} \text{ radians} = \underline{\text{______}} \text{ degrees} \quad \text{Quadrant _____} \]

\[ h) \quad \frac{7\pi}{4} \text{ radians} = \underline{\text{______}} \text{ degrees} \quad \text{Quadrant _____} \]

**AREA OF A SECTOR OF A CIRCLE**

If we go back to the idea of placing the vertex of an angle at the center of a circle (creating a central angle, as mentioned earlier), the area created by the angle is called a sector. (The shaded area in the circle on the right is a sector.) The formula for the area of a sector is $A = \frac{1}{2} r^2 \theta$, where $r$ is the radius of the circle and $\theta$ is the radian measurement of the central angle. As with the arc length formula, the central angle must be in RADIANS in order to use this formula.

**Example 5:** $A$ denotes the area of the sector of a circle or radius $r$ formed by the central angle $\theta$. Find the missing quantity. Think about the unit of Area!!

\[ a) \quad r = 6 \text{ feet}, \theta = 2 \text{ radians}, \quad A = \underline{\text{______}} \]

\[ b) \quad \theta = 120^\circ, \ r = 3 \text{ meters}, \quad A = \underline{\text{______}} \]
LINEAR SPEED OF AN OBJECT TRAVELING IN CIRCULAR MOTION

Suppose that an object moves around a circle of radius $r$ at a constant speed. If $s$ is the distance traveled in time $t$ around this circle, then the linear speed, $v$, of the object is defined as $v = \frac{s}{t}$. (Note that "$s"$ is a DISTANCE, not a speed. Students often think "$s"$ should be speed because the word speed begins with the letter "s". But you will find that in this class and your future calculus classes, "$s"$ always means either “position” or “distance”. "$v"$ – for velocity – is the letter most commonly used for speed.)

As this object travels around the circle, suppose that $\theta$ (measured in radians) is the central angle swept out over a period of time $t$. We define the angular speed, $\omega$, of the object as $\omega = \frac{\theta}{t}$.

Since we previously learned that arc length, $s$, is equal to radius times the central angle ($s = r \theta$), then we can replace $s$ in the linear speed formula with $r \theta$, to get $v = \frac{r \theta}{t}$. But since $\frac{\theta}{t}$ is equal to angular speed, $\omega$, the linear speed formula simplifies to $v = r \omega$. If we solve for $\omega$, we get $\omega = \frac{v}{r}$.

In other words, the linear speed (length per unit time) equals the radius times the angular speed (where the angular speed is measured in radians per unit time). Often the angular speed is given in revolutions per unit time, so you have to convert this to radians, remembering that $1$ revolution $= 2\pi$ radians.

Example 6:

An object is traveling around a circle with a radius of 2 meters. If the object is released and travels 5 meters in 20 seconds, what is its linear speed? What is its angular speed?

Example 7:

The radius of each wheel of a car is 15 inches. If the wheels are turning at a rate of 12 revolutions per second, how fast is the car moving? Express your answer in inches per second and also in miles per hour. (5280 feet = 1 mile)
Section 7.2 – Right Triangle Trigonometry

These are the angles we will use most often in this class (as well as in your calculus classes in the future). You need to commit these angles to memory as soon as possible!

Here is a circle showing all of these angles on it (plus a few more). The first number is the degree measurement; the second number is the radian measurement. Pay particular attention to the RADIANS measures of the STARRED angles.

If you learn well from playing games/quizzes, these websites quiz you on locating angles on a circle:

http://www.purposegames.com/game/angles-of-the-unit-circle-radians-quiz (for the radians quiz)

http://www.purposegames.com/game/angles-of-the-unit-circle-degrees-quiz (for the degrees quiz)
THE TRIGONOMETRIC FUNCTIONS

An **acute angle** is any angle that is between $0^\circ$ and $90^\circ$ (or $0$ and $\frac{\pi}{2}$ radians). Drawing a vertical line between the terminal side and the initial side of the angle creates a **right triangle**. The side labeled “$c$” is the **hypotenuse** of the right triangle. (The hypotenuse is the side directly across from the right angle.) The leg of the triangle that is directly opposite from the angle $\theta$ is called the **opposite** side. And the leg of the triangle that is **attached** to the angle $\theta$ is called the **adjacent** side.

Assuming $\theta$ is an acute angle ($\theta$ can *never* be the right angle!), we define the six trigonometric functions based on the ratios between any two sides of this triangle. The six trig functions are **Sine** ($\sin$), **Cosine** ($\cos$), **Tangent** ($\tan$), **Cosecant** ($\csc$), **Secant** ($\sec$), and **Cotangent** ($\cot$), and their definitions are given in the table below:

<table>
<thead>
<tr>
<th>Function</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>opposite ( \frac{b}{c} ) hypotenuse</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>adjacent ( \frac{a}{c} ) hypotenuse</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>opposite ( \frac{b}{a} ) adjacent</td>
</tr>
<tr>
<td>$\csc \theta$</td>
<td>hypotenuse ( \frac{c}{b} ) opposite</td>
</tr>
<tr>
<td>$\sec \theta$</td>
<td>hypotenuse ( \frac{c}{a} ) adjacent</td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td>opposite ( \frac{a}{b} ) adjacent</td>
</tr>
</tbody>
</table>

Here’s something to help you remember what sine, cosine, and tangent are: “**Soh Cah Toa**”!

- **SOH** means: $\sin = \frac{\text{Opposite}}{\text{Hypotenuse}}$
- **CAH** means: $\cos = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
- **TOA** means: $\tan = \frac{\text{Opposite}}{\text{Adjacent}}$

Then, remember:

- **Cosecant** is the reciprocal of Sine.
- **Secant** is the reciprocal of Cosine.
- **Cotangent** is the reciprocal of Tangent.

These are the **Reciprocal Identities**. Note that the “$s$” of sin pairs with the “$c$” of csc and the “$c$” of cos pairs with the “$s$” of sec. So, for instance, if you are ever trying to remember which one (sec or csc) is the reciprocal of cosine, remember that a “$c$” is always paired with an “$s$”, so it must be sec that is the reciprocal of cosine.
The Quotient Identities state:

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

You can see evidence of this using information from the table on the previous page:

\[
\sin \theta = \frac{b}{c} \quad \text{and} \quad \cos \theta = \frac{a}{c}. \quad \text{So if} \quad \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \text{then} \quad \tan \theta = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \cdot \frac{c}{a} = \frac{b}{a} \quad \text{(which matches the definition of tangent given in the red-outlined box above)}
\]

Pythagorean Identities (these come from using the Pythagorean Theorem: \(a^2 + b^2 = c^2\)):

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad \cot^2 \theta + 1 = \csc^2 \theta
\]

The first and second ones are the most important and will be used often!

Example 1:

a) \(\sin^2 \left(\frac{\pi}{6}\right) + \cos^2 \left(\frac{\pi}{6}\right)\)

b) \(\sec^2 28^\circ - \tan^2 28^\circ\)

Important Algebra Review on Rationalizing Denominators

Recall from your Intermediate and/or College Algebra class that you should never leave a radical in the denominator of a fraction. The act of getting rid of the radical is called "Rationalizing the Denominator". To rationalize a denominator when it contains a single radical, just multiply the numerator and denominator by the exact radical that is in the denominator.

Example 2: Rationalize the Denominator in each of the fractions.

a) \(\frac{2}{\sqrt{5}}\)

b) \(\frac{1}{\sqrt{2}}\)

c) \(\frac{3}{\sqrt{3}}\)
Finding the Values of the Trigonometric Functions When One Is Known

Given the value of one trigonometric function of an acute angle $\theta$, the exact value of each of the remaining five trigonometric functions of $\theta$ can be found in either of two ways.

Method 1 Using the Definition

**STEP 1:** Draw a right triangle showing the acute angle $\theta$.

**STEP 2:** Two of the sides can then be assigned values based on the value of the given trigonometric function.

**STEP 3:** Find the length of the third side by using the Pythagorean Theorem, $(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2$.

**STEP 4:** Use the definitions in equation (1) to find the value of each of the remaining trigonometric functions.

Method 2 Using Identities

Use appropriately selected identities to find the value of each of the remaining trigonometric functions.

---

Example 3: Find the value of the six trigonometric functions of the angle $\theta$ in each figure.

**a)**

[Diagram of a right triangle with sides labeled 2, 4, and 0.]

**b)**

[Diagram of a right triangle with sides labeled 1, $\sqrt{5}$, and 0.]
Example 4: Find the exact value of each of the remaining five trigonometric functions of the acute angle $\theta$.

a) $\csc \theta = 5$

b) $\sec \theta = \frac{5}{3}$

**COMPLEMENTARY ANGLE THEOREM**

The trig functions can be grouped into three pairs of cofunctions. **The cofunction pairs are 1) sine and cosine, 2) secant and cosecant, and 3) tangent and cotangent.** (Notice that the second function of each cofunction pair is just the first function with the "co-" prefix!)

Recall from previous classes that **complementary angles add up to 90°**. For example, $70^\circ$ and $20^\circ$ are complementary angles, as are $40^\circ$ and $50^\circ$.

The **Complementary Angle Theorem** states that **cofunctions of complementary angles are equal**. So, for example, $\sin(70^\circ) = \cos(20^\circ)$ and $\cot(40^\circ) = \tan(50^\circ)$.

**Example 5:** Find the exact value of each expression.

a) $\frac{\cos 40^\circ}{\sin 50^\circ}$

b) $\cot 60^\circ - \frac{\sin 30^\circ}{\sin 60^\circ}$

c) $\cot 25^\circ \cdot \csc 65^\circ \cdot \sin 25^\circ$
Section 7.3 – Computing the Values of Trigonometric Functions of Acute Angles

FIND THE EXACT VALUES OF THE TRIG FUNCTIONS OF 45° \( \left( \frac{\pi}{4} \right) \) radians

Given a right triangle with one angle of 45°, what is the measurement of the other angle? _________ What does this tell you about the lengths of the legs of the triangle (a and b)? ____________________________

So if side \( a \) has a length of 1, the length of side \( b \) is _________ and, using the Pythagorean Theorem, we find that the length of side \( c \) (the hypotenuse) is ________.

This type of triangle, which, considering the angle measurements in degrees is called a _____ - _____ - _____ triangle, is commonly used in both trigonometry and geometry. If we considered the same triangle with its angles measured in radians, it would be called a _____ - _____ - _____ triangle. You need to memorize that the legs of this type of triangle are each equal to _____ and its hypotenuse is equal to ______.

The hypotenuse of a 45-45-90 triangle is equal to \( \sqrt{2} \) times the length of its leg.

Example 1: Find the values of the following trig functions. Rationalize denominators where necessary.

a) \( \sin \left( \frac{\pi}{4} \right) \)  
b) \( \cos \left( \frac{\pi}{4} \right) \)  
c) \( \tan \left( \frac{\pi}{4} \right) \)  
d) \( \csc \left( \frac{\pi}{4} \right) \)  
e) \( \sec \left( \frac{\pi}{4} \right) \)  
f) \( \cot \left( \frac{\pi}{4} \right) \)
FIND THE EXACT VALUES OF THE TRIG FUNCTIONS OF

\[ 30^\circ \left( \frac{\pi}{6} \text{ radians} \right) \text{ AND } 60^\circ \left( \frac{\pi}{3} \text{ radians} \right) \]

Given a 30° - 60° - 90° triangle, with the hypotenuse of length \( c = 2 \) as shown in the figure to the left, what would be the length of the base \( (a) \)? We can't tell using only this triangle. But if we attached its mirror image (as shown in the figure to the right), it would create an equilateral triangle. What do you know about equilateral triangles?

So, what would be the length of the entire base of the equilateral triangle on the right? __________

Then, what would be the length of the base of the triangle in the figure on the left? ________

Now that you know the length of the hypotenuse, \( c \), and the base, \( a \), use the Pythagorean Theorem to find the length of the remaining leg, \( b \).

\[ b = \] ________

So a 30° - 60° - 90° triangle, which in radians would be called a _____ - _____ - _____ triangle, has the following side measurements:

- Short Leg = ________
- Long Leg = ________
- Hypotenuse = ________

The hypotenuse of a 30-60-90 triangle is **double** the length of its short leg.

The length of the long leg is equal to \( \sqrt{3} \) times the length of its short leg.

You need to memorize the lengths of the sides of this type of triangle!!

Recall that the short leg is adjacent to the wider 60° angle and the long leg is adjacent to the skinnier 30° angle.

Label the lengths of the sides of the triangle to the left. Then use the triangle to fill in the table below. Remember "Soh Cah Toa"!

<table>
<thead>
<tr>
<th>( \theta ) (Radians)</th>
<th>( \theta ) (Degrees)</th>
<th>sin ( \theta )</th>
<th>cos ( \theta )</th>
<th>tan ( \theta )</th>
<th>csc ( \theta )</th>
<th>sec ( \theta )</th>
<th>cot ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{6} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2: Find the exact value of each expression.

a) \[2 \sin 45^\circ + 4 \cos 30^\circ\]

b) \[\sec \left( \frac{\pi}{4} \right) + 2 \csc \left( \frac{\pi}{3} \right)\]

c) \[\sec^2 \left( \frac{\pi}{3} \right) - \tan^2 \left( \frac{\pi}{4} \right)\]

d) \[1 + \tan^2 30^\circ - \csc^2 45^\circ\]

USING A CALCULATOR TO APPROXIMATE THE VALUES OF THE TRIG FUNCTIONS OF ACUTE ANGLES

You will not be using your calculator very much in this course, but on occasion you will need to be able to find decimal approximations of the trig functions of angles (for instance, angles other than 30°, 45°, and 60°). When using your calculator, you just have to set the MODE to either radians or degrees, depending on the information given in the problem. Also, your calculator has buttons for sine, cosine, and tangent, but NOT for cosecant, secant, and cotangent. So, to enter these reciprocal functions, you have to put them in as follows: \(\csc \theta = \frac{1}{\sin \theta}\), \(\sec \theta = \frac{1}{\cos \theta}\), and \(\cot \theta = \frac{1}{\tan \theta}\).

Example 3: Use a calculator to find the approximate value of each expression. Round the answer to two decimal places.

a) \(\cos 14^\circ\) (make sure calculator is in DEGREE MODE)

b) \(\tan 1\) (make sure calculator is in RADIUS MODE)

c) \(\csc 21^\circ\) (in DEGREE MODE, enter as shown in the figure to the right)

d) \(\sec \left( \frac{\pi}{12} \right)\)
MODEL AND SOLVE APPLIED PROBLEMS INVOLVING RIGHT TRIANGLES

Example 4:
A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 37. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle β is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

Example 5:
Adorning the top of the Board of Trade building in Chicago is a statue of Ceres, the Roman goddess of wheat. From street level, two observations are taken 400 feet from the center of the building. The angle of elevation to the base of the statue is found to be 55.1° and the angle of elevation to the top of the statue is 56.5°. See Figure 36(a). What is the height of the statue?
In this section we will find the values of the trig functions of non-acute angles (angles that do not lie in Quadrant 1).

**USING COTERMINAL ANGLES TO FIND THE EXACT VALUE OF A TRIGONOMETRIC FUNCTION**

Coterminal angles have the same initial side and terminal side (they may just go through varying numbers of revolutions, or they may go in opposite directions, to get from the initial side to the terminal side).

For instance, in Figure (a) to the right, \( \alpha \) and \( \beta \) are coterminal because they have the same initial and terminal sides. **Coterminal angles do not have the same angle measurement** (we know that \( \alpha \) is a ________ angle because it goes in a counterclockwise rotation, and \( \beta \) is a ________ angle because it goes in a clockwise rotation), but they begin and end at the same place.

In Figure (a), \( \alpha \) appears to be _____ degrees (or _____ radians), and \( \beta \) appears to be _____ degrees (or _____ radians).

In Figure (b) to the left, \( \alpha \) and \( \beta \) are also coterminal angles. Again, they do not have the same angle measurement since angle \( \beta \) did one complete revolution before moving the rest of the way to the terminal side. In Figure (b), angle \( \alpha \) appears to be _______ degrees (or _______ radians), and angle \( \beta \) appears to be _______ degrees (or _______ radians).

In short, if the given angle is **positive**, find the Smallest Positive Coterminal Angle (SPCA) by **subtracting** 360° (or 2\( \pi \) radians) as many times as necessary until the answer is between 0° and 360° (or 0 and 2\( \pi \) radians).

If the given angle is **negative**, find the SPCA by **adding** 360° (or 2\( \pi \) radians) as many times as necessary until the answer is between 0° and 360° (or 0 and 2\( \pi \) radians).
Example 2:

a) \(390^\circ\) is coterminal to \(30^\circ\). Why?

b) \(\frac{-7\pi}{4}\) is coterminal to \(\frac{\pi}{4}\). Why?

c) \(-960^\circ\) is coterminal to \(_____^\circ\).

d) \(\frac{37\pi}{4}\) is coterminal to \(_____\) radians.

★Trigonometric functions of coterminal angles are equal.★

For example, \(\sin(390^\circ) = \sin(30^\circ) = \) \(____\). And \(\sec\left(\frac{-7\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) = \) \(____\)

Example 3: Use a coterminal angle to find the exact value of each expression.

a) \(\cos 405^\circ\)

b) \(\cot 780^\circ\)

c) \(\sin\left(\frac{9\pi}{4}\right)\)
DETERMINE THE SIGNS OF THE TRIGONOMETRIC FUNCTIONS OF AN ANGLE IN A GIVEN QUADRANT

Moving counterclockwise and starting in Quadrant I, label the quadrants in the figure to the right 'A' (for "All"), 'S' (Sine), 'T' (Tangent), and 'C' (Cosine). These letters represent which of the 3 main trig functions are positive in each quadrant. For instance, Quadrant III contains the letter T, so Tangent is positive there. Come up with a mnemonic of your own to help you remember the order of the letters:

____________________________________________________________________________________

FIND THE REFERENCE ANGLE OF A GENERAL ANGLE

For any angle \( \theta \), you can find an acute reference angle formed by the terminal side of \( \theta \) and the x-axis. The reference angle is the shortest distance between the terminal side of the angle and the x-axis.

- For Radian angles, just cover up the numerator coefficient, and what is left is the reference angle.
- For Degree angles, determine the distance to 180° for angles in Quadrants 2 or 3, and the distance to 360° for angles in Quadrant 4.

Example 4: Find the reference angle for the following angles.

a) \( 150° \)  
b) \( \frac{-5\pi}{6} \)  
c) \( \frac{9\pi}{4} \)

d) \( 315° \)  
e) \( 120° \)  
f) \( \frac{4\pi}{3} \)
USE A REFERENCE ANGLE TO FIND THE EXACT VALUE OF A TRIGONOMETRIC FUNCTION

The absolute values of the trig functions of a general angle \( \theta \) are equal to the values of the trig functions of its reference angle (only the sign – positive or negative – varies, depending on the quadrant the angle is in, as we saw above). Thus, as long as you know the values of the trig functions of the angles in Quadrant I, then you can figure out the values of the trig functions for all other angles by determining the reference angle and the sign (positive or negative) depending on the quadrant it is in.

Before we do the final two examples, we are going to learn how to create a table that shows sine, cosine, and tangent of 0°, 30°, 45°, 60°, and 90°. It will be very valuable for you to learn how to recreate this table for tests!!!

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0° (0 rad)</th>
<th>30° ( \left( \frac{\pi}{6} \right) )</th>
<th>45° ( \left( \frac{\pi}{4} \right) )</th>
<th>60° ( \left( \frac{\pi}{3} \right) )</th>
<th>90° ( \left( \frac{\pi}{2} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cos( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tan( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Steps to fill in the boxes**

1) Near the top of the boxes in the first row, write the numbers 0, 1, 2, 3, and 4.

2) Near the top of the boxes in the second row, write the numbers 4, 3, 2, 1, and 0.

3) Take the square root of each number you have written.

4) Divide every number by 2.

5) To fill in the tangent row, recall that \( \tan \theta = \frac{\sin \theta}{\cos \theta} \), so divide the number in the first box by the number in the second box and that will give you the tangent value.

**Example 5:** Use the reference angle to find the exact value of each expression.

a) \( \cos 210° \) 

b) \( \sin 120° \)
Example 5 (continued): Use the reference angle to find the exact value of each expression.

c) \( \sin (-240^\circ) \) 

d) \( \sec (300^\circ) \)

e) \( \cos \left( \frac{2\pi}{3} \right) \) 

f) \( \csc \left( \frac{7\pi}{4} \right) \)

g) \( \tan \left( \frac{8\pi}{3} \right) \) 

h) \( \cot \left( -\frac{\pi}{6} \right) \)

i) \( \sec \left( \frac{11\pi}{4} \right) \) 

j) \( \sec (-225^\circ) \)
Example 6: Find the exact value of each of the remaining trigonometric functions of \( \theta \).

a) \( \cos \theta = \frac{3}{5} \), \( \theta \) in Quadrant IV

b) \( \sin \theta = \frac{5}{13} \), \( 90^\circ < \theta < 180^\circ \)

c) \( \csc \theta = 3 \), \( \cot \theta < 0 \)

d) \( \sin \theta = \frac{2}{3} \), \( \pi < \theta < \frac{3\pi}{2} \)
A unit circle is a circle with radius = 1 whose center is at the origin. Since we know that the formula for the circumference of a circle is \( C = 2\pi r \), then for the unit circle, the circumference would be \( C = 2\pi(1) = 2\pi \). Thus, one complete revolution around the unit circle moves \( 2\pi \) units.

**FIND THE EXACT VALUES OF THE TRIGONOMETRIC FUNCTIONS USING THE UNIT CIRCLE**

For any point \( P = (a, b) \) that lies on the unit circle and creates an angle \( \theta \) with the positive x-axis, the x-value (a) is the value of \( \cos \theta \) and the y-value (b) is the value of \( \sin \theta \).

This is super-duper important so I'm going to repeat it: for any point that lies on the unit circle, its x-coordinate is its cosine value and its y-coordinate is its sine value.

We know from the Quotient Identities that \( \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y \text{- coordinate}}{x \text{- coordinate}} \), and \( \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{x \text{- coordinate}}{y \text{- coordinate}} \).

Also, from the Reciprocal Identities, \( \csc \theta = \frac{1}{\sin \theta} = \frac{1}{y \text{- coordinate}} \) and \( \sec \theta = \frac{1}{\cos \theta} = \frac{1}{x \text{- coordinate}} \).

**KNOW THE DOMAIN AND RANGE OF THE TRIGONOMETRIC FUNCTIONS**

The domain of a function is the set of all real number inputs that give real number outputs. For polynomial functions, the domain is \( \mathbb{R} \), and for rational functions, the domain is all real numbers except \( \text{______________________________} \).

Example 1: Find the domain for each trig function of \( \theta \). (In other words, find all the values of \( \theta \) that cause the function to be undefined. Then exclude those values from the domain.)

a) \( \sin \theta \)  

b) \( \cos \theta \)

c) \( \tan \theta \)  

d) \( \csc \theta \)

e) \( \sec \theta \)  

f) \( \cot \theta \)
Now let's examine the ranges of the trig functions, using the figure on the right.

Recall that \( \cos \theta \) is represented by the \textbf{x-values} as we move around the unit circle. The left-most (and thus smallest) \textbf{x-value} is negative one \((-1)\), and the right-most (and thus largest) \textbf{x-value} is positive one \((1)\). Thus, the range for \( \cos \theta \) is \([-1, 1]\) \((\text{or } -1 \leq \cos \theta \leq 1)\). Similarly, recall that \( \sin \theta \) is represented by the \textbf{y-values} as we move around the unit circle. The bottom-most (smallest) \textbf{y-value} is \(-1\) and the top-most (largest) \textbf{y-value} is \(1\). So the range for \( \sin \theta \) is also \([-1, 1]\) \((\text{or } -1 \leq \sin \theta \leq 1)\).

The rest of the ranges are not quite as obvious to find, so I will list them below.

\[
\begin{align*}
\csc \theta & \leq -1 \quad \text{or} \quad \csc \theta \geq 1 \\
\sec \theta & \leq -1 \quad \text{or} \quad \sec \theta \geq 1
\end{align*}
\]

The values of \( \csc \theta \) are either less than (or equal to) \(-1\) or greater than (or equal to) \(1\).

The values of \( \sec \theta \) are either less than (or equal to) \(-1\) or greater than (or equal to) \(1\).

\[-\infty < \tan \theta < \infty \quad \text{and} \quad -\infty < \cot \theta < \infty\]

The range of values that both \( \tan \theta \) and \( \cot \theta \) can take is All Real Numbers.

The table below summarizes the Domains and Ranges for all six trigonometric functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Symbol</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine ( f(\theta) = \sin \theta )</td>
<td>All real numbers</td>
<td>All real numbers from (-1) to (1), inclusive</td>
<td></td>
</tr>
<tr>
<td>cosine ( f(\theta) = \cos \theta )</td>
<td>All real numbers</td>
<td>All real numbers from (-1) to (1), inclusive</td>
<td></td>
</tr>
<tr>
<td>tangent ( f(\theta) = \tan \theta )</td>
<td>All real numbers, except odd multiples of (\frac{\pi}{2}) ((90^\circ))</td>
<td>All real numbers</td>
<td></td>
</tr>
<tr>
<td>cosecant ( f(\theta) = \csc \theta )</td>
<td>All real numbers, except integer multiples of (\pi) ((180^\circ))</td>
<td>All real numbers greater than or equal to (1) or less than or equal to (-1)</td>
<td></td>
</tr>
<tr>
<td>secant ( f(\theta) = \sec \theta )</td>
<td>All real numbers, except odd multiples of (\frac{\pi}{2}) ((90^\circ))</td>
<td>All real numbers greater than or equal to (1) or less than or equal to (-1)</td>
<td></td>
</tr>
<tr>
<td>cotangent ( f(\theta) = \cot \theta )</td>
<td>All real numbers, except integer multiples of (\pi) ((180^\circ))</td>
<td>All real numbers</td>
<td></td>
</tr>
</tbody>
</table>
USE THE PERIODIC PROPERTIES TO FIND THE EXACT VALUES OF THE TRIGONOMETRIC FUNCTIONS

Trigonometric functions are called **periodic functions**. A function \( f \) is called periodic if there is a positive number \( p \) such that \( f(\theta + p) = f(\theta) \) whenever \( \theta \) and \( \theta + p \) are both in the domain of \( f \). The smallest value that \( p \) can take is called the **fundamental period** of \( f \). For sine and cosine (and thus also their reciprocal functions, __________ and __________, respectively), the fundamental period is __________.

So \( \sin (\theta + 2 \pi k) = \sin \theta \) and \( \cos (\theta + 2 \pi k) = \cos \theta \), where \( k \) is any integer (and the same for their reciprocals).

Tangent and cotangent have a fundamental period (also just called a "period") of \( \pi \). Therefore, \( \tan (\theta + \pi k) = \tan \theta \) and \( \cot (\theta + \pi k) = \cot \theta \).

Evaluating trigonometric functions using periodicity really just amounts to finding the Smallest Positive Coterminal Angle, as we learned in section 7.4.

USE EVEN-ODD PROPERTIES TO FIND THE EXACT VALUES OF THE TRIGONOMETRIC FUNCTIONS

You learned in college algebra (and again in precalculus) that an **even** function is one where \( f(-x) = f(x) \), and that an **odd** function is one where \( f(-x) = -f(x) \).

Look at the figure to the left. Recall that for the point \( P = (a, b) \), \( a \) is \( \cos \theta \) and \( b \) is \( \sin \theta \). Now look at its mirror image point \( Q = (a, -b) \) (this is just the point \( P \) flipped down across the x-axis). So the angle between the positive x-axis and the point \( Q \) is equal to \(-\theta\). Using these points, we can see that:

\[ \cos(-\theta) = ____ \], which is the __________ as \( \cos(\theta) \), but

\[ \sin(-\theta) = ____ \], which is _________________ of \( \sin(\theta) \).

Therefore, \( \cos(-\theta) = \cos(\theta) \), so cosine is an ______________ function, but \( \sin(-\theta) = ____ \), so sine is an ______________ function.

It turns out that cosine (and its reciprocal function, __________) are the only __________ functions.

All of the remaining functions (__________, __________, __________, and __________) are ______.
Important Note: The Even-Odd properties are to be used ONLY if the negative sign is INSIDE the parentheses (in the argument of the trig function).

Even function → Erase the negative sign

Odd function → bring the negative Outside

Example 2: Use the even-odd properties to rewrite the following trig expressions. Do not find the exact values of the expressions at this time. (Hint: if the function is even, just erase the negative sign. If the function is odd, then move the negative sign out in front of the trig function.)

a) \( \cos(-30^\circ) \)  
b) \( \sin(-135^\circ) \)

c) \( \sec(-270^\circ) \)  
d) \( \sin\left(-\frac{\pi}{3}\right) \)

e) \( \sec(-\pi) \)  
f) \( \cot\left(-\frac{\pi}{4}\right) \)
THE UNIT CIRCLE

The unit circle is a very valuable tool. You can see at a glance the cosine and sine values for the four quadrantal angles as well as the three main angles within each quadrant (remember, \( \cos \theta = x\)-value and \( \sin \theta = y\)-value). We are going to fill in the unit circle below. We will put the angle measurements in both degrees and radians, then fill in the cosine and sine values for each angle.
Example 3: Looking at the unit circle you created on the previous page, you can quickly find the following values:

a) \( \cos \left( \frac{7\pi}{6} \right) = \underline{___________} \)  \hspace{1cm} b) \( \sin \left( \frac{2\pi}{3} \right) = \underline{___________} \)

b) \( \sin \left( \frac{11\pi}{6} \right) = \underline{___________} \)  \hspace{1cm} d) \( \cos \left( \frac{5\pi}{4} \right) = \underline{___________} \)

You can also quickly identify the values of secant and cosecant, since these are just the \underline{___________________} of the cosine and sine values, respectively. For example, looking at the unit circle, you can quickly find the following values:

\[ c) \ \csc \left( \frac{5\pi}{6} \right) \]  \hspace{1cm} \[ f) \ \sec \left( \frac{4\pi}{3} \right) \]

\[ g) \ \sec \left( \frac{7\pi}{4} \right) \]  \hspace{1cm} \[ h) \ \csc \left( \frac{5\pi}{3} \right) \ \text{(rationalize the denominator!)} \]

And with a little bit of calculation effort, we can also relatively easily find the values of tangent \( \left( \frac{y}{x} \right) \) and cotangent \( \left( \frac{x}{y} \right) \):

\[ i) \ \tan \left( \frac{3\pi}{4} \right) \]  \hspace{1cm} \[ j) \ \cot \left( \frac{\pi}{6} \right) \]

Example 4: Now Return to Example 2 on page 4 and find the \textbf{exact values} of the expressions.
Quadrantal Angles

\[ \theta = 0^\circ = 360^\circ = 0 \text{ radians} = 2\pi \text{ radians: corresponds to the point (1, 0) on the unit circle. Thus, } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

\[ \theta = 90^\circ = \frac{\pi}{2} \text{ radians: corresponds to the point (0, 1) on the unit circle. Thus, } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

\[ \theta = 180^\circ = \pi \text{ radians: corresponds to the point (-1, 0) on the unit circle. Thus, } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

\[ \theta = 270^\circ = \frac{3\pi}{2} \text{ radians: corresponds to the point (0, -1) on the unit circle. Thus, } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

Quadrant I Angles

\[ \theta = 30^\circ = \frac{\pi}{6} \text{ radians: } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

\[ \theta = 45^\circ = \frac{\pi}{4} \text{ radians: } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

\[ \theta = 60^\circ = \frac{\pi}{3} \text{ radians: } \cos \theta = \underline{\phantom{0}}, \sin \theta = \underline{\phantom{0}}. \]

Example 5: Find the exact value of each trigonometric function. Use the even-odd properties whenever the argument is negative.

a) \( \sin (-45^\circ) \)

b) \( \cos (-30^\circ) \)

c) \( \sec \left( \frac{2\pi}{3} \right) \)

d) \( \csc \left( \frac{4\pi}{3} \right) \)

e) \( \cos \left( \frac{7\pi}{4} \right) \)

f) \( \tan \left( \frac{11\pi}{6} \right) \)
Section 7.6 - Graphs of the Sine and Cosine Functions

We are going to learn how to graph the sine and cosine functions on the xy-plane. Just like with any other function, it is easy to do by plotting points. For instance, plot points to graph the line \( y = 3x - 1 \):

<table>
<thead>
<tr>
<th>x</th>
<th>y = 3x - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

We will need to use the known trig values of our common angles to fill in the x-y table to help us graph sine and cosine. So let's review our unit circle here:

First let's look at \( f(x) = \sin(x) \). We could choose any x-value (where the x-values are the ANGLES) to help us plot points, but to make our lives as easy as possible, we should use angles that give us easy-to-plot y-values. To determine which angles to use, let's find

\[
f(0) = \fbox{} , \quad f\left(\frac{\pi}{6}\right) = \fbox{} ,
\]

\[
f\left(\frac{\pi}{4}\right) = \fbox{} , \quad f\left(\frac{\pi}{3}\right) = \fbox{} , \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \fbox{} .
\]

Knowing that \( y = f(x) \), which of these angles gives us the easiest-to-plot y-values?

Now let's go through the same process for cosine. Let \( f(x) = \cos(x) \). Find \( f(0) = \fbox{} , \quad f\left(\frac{\pi}{6}\right) = \fbox{} , \quad f\left(\frac{\pi}{4}\right) = \fbox{} , \quad f\left(\frac{\pi}{3}\right) = \fbox{} , \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \fbox{} . \) Which of these angles gives us the easiest-to-plot y-values?
THE GRAPH OF THE SINE FUNCTION

To graph \( y = \sin x \), we are going to plot points using all of the quadrant angles \((0, \pi/2, \pi, \text{ and } 3\pi/2)\), plus the angles from all four quadrants that have \(\pi/6\) as their reference angle. Recall that \(\sin(x)\) is positive in quadrants _____ and _____ and it is negative in quadrants _____ and _____.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = \sin x)</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi/6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi/2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5\pi/6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7\pi/6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3\pi/2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(11\pi/6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2\pi)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We know that sine is a periodic function, so although we only graphed it from 0 to \(2\pi\), the graph actually repeats forever in both directions. The period of sine is \(2\pi\), so the graph does one complete cycle over that period, then it starts over.

What are the \(x\)-intercepts of \(\sin(x)\)?

The maximum \(y\)-value is _____ and the minimum \(y\)-value is _____.

To see how the unit circle ties in with the graph of \(\sin(x)\), click here:

[http://www.ies.co.jp/math/java/samples/graphSinX.html](http://www.ies.co.jp/math/java/samples/graphSinX.html)
THE GRAPH OF THE COSINE FUNCTION

Now we are going to follow the same basic steps to plot the graph of \( y = \cos x \). We are going to use the quadrantal angles again, plus the angles from all four quadrants that have \( \frac{\pi}{3} \) as their reference angle. Recall that \( \cos(x) \) is **positive** in quadrants _____ and _____ and it is **negative** in quadrants _____ and _____.

We know that cosine is a periodic function, so although we only graphed it from 0 to \( 2\pi \), the graph actually repeats forever in both directions.

You may recall learning about transformations of functions in your college algebra and precalculus courses. Remember that \( f(x) + c \) moves a function up \( c \) units, \( f(x) - c \) moves it down \( c \) units, \( f(x + c) \) moves it left \( c \) units, and \( f(x - c) \) moves it right \( c \) units. So, if we graphed \( \cos \left( x - \frac{\pi}{2} \right) \), we would expect it to take the graph of \( \cos x \) and move it to the ______ by \( \pi/2 \) units. Sketch this transformed graph on the figure above.

What function does it look like? ______________________________
As it turns out, \[
\sin x = \cos \left( x - \frac{\pi}{2} \right)
\]

To see this, plot the following in your graphing calculator in RADIAN mode:

How many graphs do you see? ______

Because of this relationship between the graphs of sine and cosine, we say that the graph of all functions of the form \( y = A \sin(\omega x) + B \) and \( y = A \cos(\omega x) + B \) are **sinusoidal graphs**.

**AMPLITUDE**

The **amplitude** \(|A|\) of a sinusoidal graph is the distance between the vertical center of the graph and its highest (maximum) and lowest (minimum) points. Both of the graphs we looked at previously (\( y = \sin x \) and \( y = \cos x \)) had vertical centers at \( y = 0 \) and amplitudes of \( A = _____ \). When looking at a graph, you can calculate the amplitude as follows:

\[
A = \frac{\text{max} - \text{min}}{2}
\]

You can think of the “vertical center” as the horizontal line that cuts the graph in half (i.e. half the graph lies above it and half the graph lies below it). To find the vertical center (if you can’t tell what it is just by looking at the graph), it is always at: \( y = \text{max} - A \) (or \( y = \text{min} + A \)).

Look at the four figures below. Determine their vertical centers (it may help to draw them in) and find their amplitudes.

Amplitude is _______. Centered at \( y = _____ \).  
Amplitude is _______. Centered at \( y = _____ \).

Amplitude is _______. Centered at \( y = _____ \).  
Amplitude is _______. Centered at \( y = _____ \).
PERIOD

As we learned in section 7.5, the trigonometric functions are periodic functions, and \( y = \sin x \) and \( y = \cos x \) have a period of \( 2\pi \). However, if there is a number multiplied by the \( x \), that changes the period of the function. In the case of the general sinusoidal function, \( y = A \sin(\omega x) \) and \( y = A \cos(\omega x) \), the "\( \omega \)" alters the period. To find the period, \( T \), of the sine or cosine function, apply the following formula:

\[
\text{Period} = T = \frac{2\pi}{\omega}
\]

For example, the period of \( y = \cos(4x) \) is \( T = \frac{2\pi}{4} = \frac{\pi}{2} \). This means that one complete cycle of the cosine graph would be completed between 0 and \( \frac{\pi}{2} \) (rather than the usual 0 to \( 2\pi \)). The graph of \( y = \cos(4x) \) is shown in the figure on the right. (The vertical bars indicate the start and end of one complete cycle.) If \( \omega \) is negative, then we use the even-odd properties of the functions to bring the negative sign out. Recall that \( \sin(-x) = -\sin(x) \) while \( \cos(-x) = \cos(x) \). A negative sign in front of the function indicates a reflection (flip) across the \( x \)-axis.

**Example 1:** Determine the amplitude and period of each function and state if the graph is reflected across the \( x \)-axis.

a) \( y = \sin\left(-\frac{1}{2}x\right) \)  
Amplitude: ________  
Period: _______  
Reflected (Y or N):_______

b) \( y = \frac{4}{3} \sin\left(\frac{2}{3}x\right) \)  
Amplitude: ________  
Period: _______  
Reflected (Y or N):_______

c) \( y = \frac{9}{5} \cos\left(-\frac{3\pi}{2}x\right) \)  
Amplitude: ________  
Period: _______  
Reflected (Y or N):_______

d) \( y = 3\cos(\pi x) \)  
Amplitude: ________  
Period: _______  
Reflected (Y or N):_______
GRAPHING SINUSOIDAL FUNCTIONS USING KEY POINTS

The "key points" of the sine and cosine functions are the x-values (the angles) that correspond to the max, min, and the points that lie on the vertical center. The key points for sin(x) and cos(x) are shown in the figures below.

The vertical center will be the x-axis unless there is a number added to or subtracted from the function. For instance, $y = 3\cos(2x) + 4$ is being shifted up by 4 so its vertical center is at $y = 4$. And $y = \sin\left(\frac{\pi}{4}x\right) - 1$ is being shifted down by 1 so its vertical center is at $y = -1$. If you don’t see a number added to or subtracted from the function (outside of the parentheses) then the function is not undergoing a vertical shift so the vertical center is along the x-axis (it is at $y = 0$).

Determining the vertical center is an important part of the graphing process, so in the steps below, I have added this as Step 0 at the beginning.

---

**Steps for Graphing a Sinusoidal Function of the Form**

$y = A \sin(\omega x)$ or $y = A \cos(\omega x)$ Using Key Points

**Step 0:** Determine the vertical center of the graph.

**STEP 1:** Use the amplitude $A$ to determine the maximum and minimum values of the function. This sets the scale for the y-axis.

**STEP 2:** Use the period $\frac{2\pi}{\omega}$ and divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

**STEP 3:** Use the endpoints of these subintervals to obtain five key points on the graph.

**STEP 4:** Connect these points with a sinusoidal graph to obtain the graph of one cycle and extend the graph in each direction to make it complete.
Example 2: Graph the function using key points. Show at least two cycles.

\[ y = 2 \cos \left( \frac{1}{4} x \right) \]

Vertical Center: _______  Amplitude: _____  Max: _____  Min: _____

Period: _______  Period ÷ 4: _______ (call this \( \delta \))

The 1st \( x \)-value is always 0 (unless there is a phase shift, which we will learn about in section 7.8).
Each additional \( x \)-value is the previous one plus \( \delta \).
The last \( x \)-value will be the same as the period (if not, you screwed something up!).

\textbf{x-values of Key Points:}

\[ \frac{0}{x_1}; \quad x_1 + \delta = \boxed{\text{_______}}; \quad x_2 + \delta = \boxed{\text{_______}}; \quad x_3 + \delta = \boxed{\text{_______}}; \quad x_4 + \delta = \boxed{\text{_______}}; \quad x_5 \text{ (same as } T) \]

Recall that for \( y = \cos(x) \), the \( y \)-values of the key points are 1, 0, -1, 0, 1 (you can find these by starting at \( \theta = 0 \) and traveling around the unit circle reading off the cosine values). Because the coefficient of the function is _____, you just have to multiply each \( y \)-value above by _____ to get the \( y \)-values of the key points.

\textbf{Key Points:} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}} \quad \boxed{\text{__________}}
Example 3: Graph the function using key points. Show at least two cycles.

\[ y = 4 \sin \left( \frac{\pi}{2} x \right) - 2 \]

Vertical Center: \( \underline{\text{________}} \)  
Amplitude: \( \underline{\text{______}} \)  
Max: \( \underline{\text{______}} \)  
Min: \( \underline{\text{______}} \)  

Period: \( \underline{\text{______}} \)  
\( \text{Period} \div 4: \underline{\text{______}} \) (\( \delta \))

Remember that the 1st x-value is 0 and the last x-value will be the same as the period (T).

\textbf{x-values of Key Points:}

\[ \frac{x_1}{0} ; \ x_1 + \delta = \underline{\text{________}} ; \ x_2 + \delta = \underline{\text{________}} ; \ x_3 + \delta = \underline{\text{________}} ; \ x_4 + \delta = \underline{\text{________}} \]

\( x_5 \) (same as \( T \))

Recall that for \( y = \sin(x) \), the y-values of the key points are 0, 1, 0, -1, 0 (\textit{you can find these by starting at } \theta = 0 \textit{ and traveling around the unit circle reading off the sine values}). Because the coefficient of the function is \( \underline{\text{______}} \), you need to multiply each y-value above by \( \underline{\text{______}} \), then \( \underline{\text{__________________________}} \) (to account for the vertical shift) to get the y-values of the key points.

\textbf{Key Points:} \underline{\text{__________________________}} \underline{\text{__________________________}} \underline{\text{__________________________}}
Example 4: Graph the function using key points. Show at least two cycles.

\[ y = 2 - 3\cos(3x) \]

Rewrite this as: ___________________

Vertical Center: ________  Amplitude: _____  Max: _____  Min: _____

Period: ________  Period ÷ 4: _____ (\(\delta\))

The 1st \(x\)-value is always 0. Remember, the last \(x\)-value will be the same as the period.

\[ x\text{-values of Key Points:} \quad 0; \quad x_1 + \delta = \ldots; \quad x_2 + \delta = \ldots; \quad x_3 + \delta = \ldots; \quad x_4 + \delta = \ldots; \quad x_5 \text{ (same as T)} \]

Recall that for \(y = \cos(x)\), the \(y\)-values of the key points are ______________________.

Because the coefficient of the function is _____, you need to multiply each \(y\)-value above by ______, then ____________________________ (to account for the vertical shift) to get the \(y\)-values of the key points.

Key Points:  ____________  ____________  ____________  ____________  ____________

\[ \begin{array}{c}
\begin{array}{c}
0 \\
\hline
x_1 \\
\hline
x_2 \\
\hline
x_3 \\
\hline
x_4 \\
\hline
x_5 \text{ (same as T)}
\end{array}
\end{array} \]
Example 5: Write the equation of a sine function with a vertical center at \( y = 0 \), an Amplitude of 2, and a Period of \( 4\pi \).

When asked to find the equation of a trig function given its graph, the first step is to determine if you are looking at a sine or cosine curve. To make that decision, find the point of intersection of the Vertical Center and the \( y \)-axis.

- If the graph goes through that point, it is a sine curve.
  - If it is increasing (heading uphill) through that point, it is a positive sine curve.
  - If it is decreasing (heading downhill) through that point, it is a negative sine curve.
- If the graph doesn't go through that point, it is a cosine curve.
  - If it is decreasing as it crosses the \( y \)-axis, it is a positive cosine curve.
  - If it is increasing as it crosses the \( y \)-axis, it is a negative cosine curve.

Example 6: Find an equation for each graph. Recall that \( A = \frac{(\text{max} - \text{min})}{2} \), \( \text{VC} = \text{max} - A \), and \( T = \frac{2\pi}{\omega} \) (thus \( \omega = \frac{2\pi}{T} \)).

Now try to find the equation for each graph shown on Page 4.
Section 7.7 – Graphs of the Tangent, Cotangent, Secant, and Cosecant Functions

In this section we will find the graphs of the remaining 4 trigonometric functions, starting with the tangent function.

GRAPHING TANGENT

We saw in a previous section that tangent has a period of \( \pi \) (as opposed to sine and cosine, which both have a period of \( \frac{\pi}{2} \)). Recall from our Quotient Identities that \( \tan \theta = \frac{\sin \theta}{\cos \theta} \). Thus, tangent would be undefined whenever its denominator is zero (\( \cos \theta = 0 \)), which is all odd multiples of \( \frac{\pi}{2} \). Since tangent has a period of \( \pi \) and is undefined at \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \), we will put vertical asymptotes at those values and then plot points between the asymptotes to find the graph of one cycle of tangent. Use the points given in the table to plot one cycle of tangent on the axes on the right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \tan x )</th>
<th>(( x, y ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{\pi}{3})</td>
<td>(-\sqrt{3} \approx -1.73)</td>
<td>(-\frac{\pi}{3}, -\sqrt{3})</td>
</tr>
<tr>
<td>(-\frac{\pi}{4})</td>
<td>(-1)</td>
<td>(-\frac{\pi}{4}, -1)</td>
</tr>
<tr>
<td>(-\frac{\pi}{6})</td>
<td>(-\frac{\sqrt{3}}{3} \approx -0.58)</td>
<td>(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3})</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>(\frac{\pi}{6})</td>
<td>(\frac{\sqrt{3}}{3} \approx 0.58)</td>
<td>(\frac{\pi}{6}, \frac{\sqrt{3}}{3})</td>
</tr>
<tr>
<td>(\frac{\pi}{4})</td>
<td>1</td>
<td>(\frac{\pi}{4}, 1)</td>
</tr>
<tr>
<td>(\frac{\pi}{3})</td>
<td>(\sqrt{3} \approx 1.73)</td>
<td>(\frac{\pi}{3}, \sqrt{3})</td>
</tr>
</tbody>
</table>

Properties of the Tangent Function

1. The domain is the set of all real numbers, except odd multiples of \( \frac{\pi}{2} \).
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The tangent function is periodic, with period \( \pi \).
5. The \( x \)-intercepts are \( \ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots \); the \( y \)-intercept is 0.
6. Vertical asymptotes occur at \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \)
GRAPHING COTANGENT

Recall from our Quotient Identities that \( \cot \theta = \frac{\cos \theta}{\sin \theta} \). Thus, cotangent would be undefined whenever its denominator is zero (\( \sin \theta = 0 \)), which is at all multiples of \( \pi \). Since cotangent has a period of \( \pi \) and is undefined at ________ and __________, we will put vertical asymptotes at those values and then plot points between the asymptotes to find the graph of one cycle of cotangent. **Use the points given in the table to plot one cycle of cotangent on the axes on the right.**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \cot x )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/6 )</td>
<td>( \sqrt{3} )</td>
<td>( (\pi/6, \sqrt{3}) )</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>1</td>
<td>( (\pi/4, 1) )</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>( \sqrt{3}/3 )</td>
<td>( (\pi/3, \sqrt{3}/3) )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
<td>( (\pi/2, 0) )</td>
</tr>
<tr>
<td>( 2\pi/3 )</td>
<td>( -\sqrt{3}/3 )</td>
<td>( (2\pi/3, -\sqrt{3}/3) )</td>
</tr>
<tr>
<td>( 3\pi/4 )</td>
<td>-1</td>
<td>( (3\pi/4, -1) )</td>
</tr>
<tr>
<td>( 5\pi/6 )</td>
<td>( -\sqrt{3} )</td>
<td>( (5\pi/6, -\sqrt{3}) )</td>
</tr>
</tbody>
</table>

Draw several cycles of the graphs of the tangent (left) and cotangent (right) functions below. Start by filling in the vertical asymptotes and x-intercepts then just fill in the general shapes of the graphs.
GRAPHING COSECANT

Since \( \csc x = \frac{1}{x} \), it will have vertical asymptotes wherever ________________. Thus, it will have VA's in the same spots as the ________________ function. And since the only way a rational function can equal zero is if the numerator equals zero, and the numerator of the cosecant function is 1 (with no \( x \)), then the function can never equal zero. Thus, it has NO _________________. This creates a unique graph with local minimum values where sine had maximum values and vice versa, and vertical asymptotes where sine had \( x \)-intercepts, as seen below.

The figure below shows the graph of cosecant without sine (or the vertical asymptotes) drawn in.
**GRAPHING SECANT**

Since \( \sec x = \frac{1}{\cos x} \), it will have vertical asymptotes wherever \( \cos x = 0 \). Thus, it will have VA's in the same spots as the cosine function. Just like cosecant, secant can never equal zero and thus has no x-intercepts. The graph of secant has local minimum values where cosine had maximum values and vice versa, and vertical asymptotes where cosine had x-intercepts, as seen below.

The figure below shows the graph of secant without cosine (or the vertical asymptotes) drawn in.
Example 3: Graph the functions.

a) \( y = \frac{1}{2} \csc(2x) \)

b) \( y = -3 \sec\left(\frac{x}{2}\right) \)

c) \( y = 3 \csc\left(\frac{1}{2}x\right) - 1 \)
Section 7.8 - Phase Shifts

As we discussed in Section 7.6, \( f(x + c) \) moves a function to the left \( c \) units, and \( f(x - c) \) moves it to the right \( c \) units. Notice that these horizontal shifts, which are called phase shifts, are illogical in the sense that when you see a "+" you move left and when you see a "-" you move right.

The function \( y = \sin(\omega x - \phi) \) can be rewritten as \( y = \sin(\omega(x - \frac{\phi}{\omega})) \) just by factoring the omega out. This way we can see how far to shift the function's graph \( \left( \frac{\phi}{\omega} \right) \).

Example 1: Graph \( y = 3\cos(2x + \pi) \). First, we need to factor out the 2 and rewrite it as \( y = 3\cos\left(2\left(x + \frac{\pi}{2}\right)\right) \).

This way we can see that the graph will be shifted \( \frac{\pi}{2} \) units, and because it is Plus \( \frac{\pi}{2} \), we shift to the LEFT. Following our format for graphing from section 7.6, let's fill in these blanks:

Vertical Center: _______  Amplitude: _____  Max: _____  Min: _____
Period: _______  \( \delta: _____ \)

The 1st x-value is 0 when there is no phase shift. But since this function has a phase shift, the first x-value has to be shifted \( \frac{\pi}{2} \) units to the LEFT, so \( 0 - \frac{\pi}{2} = -\frac{\pi}{2} \). The last x-value should be \( T - \frac{\pi}{2} = \pi - \frac{\pi}{2} = \frac{\pi}{2} \).

x-values of Key Points:

\( x_1 \); \( x_1 + \delta = \) _______; \( x_2 + \delta = \) _______; \( x_3 + \delta = \) _______; \( x_4 + \delta = \) _______; \( x_5 \)

For \( y = \cos(x) \), the y-values of the key points are _____________________________.

Because the coefficient of the function is _____, you need to multiply each y-value above by _____.
No addition or subtraction of the y-values is required since this function has no vertical shift.

Key Points:

________________
________________
________________
________________
________________
Example 2: Graph \( y = -2\sin(4x - \pi) - 3 \).

Factor out the \( \pi \) and re-write the function as ________________________________.

Notice there is a negative coefficient of the trig function so we know this graph will be flipped across the x-axis.

Vertical Center: _________ Amplitude: ______ Max: ______ Min: ______

Period: ______ Period ÷ 4: ______ (\( \delta \))

Phase Shift (distance and direction): ________________________________

The 1st \( x \)-value is 0 ± (phase shift). The last \( x \)-value is \( T \pm \) (phase shift).

**x-values of Key Points:**

\[
\begin{align*}
&x_1, \ x_1 + \delta = \underline{\quad}; \ x_2, \ x_2 + \delta = \underline{\quad}; \ x_3, \ x_3 + \delta = \underline{\quad}; \ x_4, \ x_4 + \delta = \underline{\quad}; \\
&x_5
\end{align*}
\]

For \( y = \sin(x) \), the \( y \)-values of the key points are _________________________________. Because the coefficient of the function is _____, you need to multiply each \( y \)-value above by _____, then __________________________ (to account for the vertical shift) to get the \( y \)-values of the key points.

**Key Points:**

________________, __________________, __________________, ________________, ________________
8.1 – The Inverse Sine, Cosine, and Tangent Functions

You learned about inverse functions in both college algebra and precalculus. The main characteristic of inverse functions is that composing one within the other always equals “x”. In mathematical notation, this is written as follows:

\[ f \left( f^{-1}(x) \right) = x \quad \text{and} \quad f^{-1} \left( f(x) \right) = x. \]

For instance, if \( f(x) = 3x + 7 \), then \( f^{-1}(x) = \frac{x - 7}{3} \). Let’s prove that these are inverse functions:

You may also remember that the domain (x-values) of \( f(x) \) is the same as the range (y-values) of \( f^{-1}(x) \), and vice versa. Thus, you can get points on one graph just by interchanging the x and y values from the other. This results in graphs that are symmetric across the line \( y = x \). The graphs of \( f(x) = 3x + 7 \) and

\[ f^{-1}(x) = \frac{x - 7}{3} \]

are shown in the figure to the right.

Notice that they are mirror images across the line \( y = x \).

Notice also that \( f(x) \) contains the points \((-2, 1)\) and \((-1, 4)\), and that \( f^{-1}(x) \) contains the points _________ and ___________.

In order to have an inverse, a function must be one-to-one, which means that its graph must pass the Horizontal Line Test. If a function is not one-to-one, it is usually possible to restrict its domain to make it one-to-one. Think of the graphs of the trig functions, such as sine, cosine, and tangent. Are these functions one-to-one? __________ In order to have inverse trig functions, we must restrict the domains of the functions to make them one-to-one. We will do this as follows: for \( \sin(x) \), restrict the domain to \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \), and for \( \tan(x) \), restrict the domain to \( -\frac{\pi}{2} < x < \frac{\pi}{2} \) (so we are only looking at sine and tangent for angles in Quadrants _______ and _______ ). For \( \cos(x) \), restrict the domain to \( 0 \leq x \leq \pi \) (so we are only looking at cosine for angles in Quadrants _______ and _______ ). These restricted domains become the _____________________ of the inverse trig functions.
Now that we have restricted the domains of sine, cosine, and tangent, we are ready to define their inverse functions.

\[ y = \sin^{-1} x \quad \text{means} \quad x = \sin y \]
where \(-1 \leq x \leq 1\) and \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\)

\[ y = \cos^{-1} x \quad \text{means} \quad x = \cos y \]
where \(-1 \leq x \leq 1\) and \(0 \leq y \leq \pi\)

Notice that the RANGE of the inverse trig function is the same as the restricted DOMAIN of the original trig function.

**FINDING THE EXACT VALUE OF INVERSE TRIGONOMETRIC FUNCTIONS**

The answer to an inverse trig problem is an \textbf{angle} — all you have to do to figure out the answer is to rearrange the problem. For instance, to find the solution to \( \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) \). the question you really ask yourself is this: Sine of what \textbf{angle} (where the angle has to be between _____ and _____) equals \( \frac{\sqrt{2}}{2} \)? Since sine is negative for angles in quadrant _____, we know our angle has to come from quadrant ______. So what is the angle in quadrant _____ whose sine equals \( \frac{\sqrt{2}}{2} \)? Answer: ____________
Let’s look at another one: \( \cos^{-1} \left( -\frac{1}{2} \right) \). The question is this: Cosine of what angle (between _____ and ______) equals \(-\frac{1}{2}\)? Since cosine is negative for angles in quadrant _______, we know our answer has to come from that quadrant. So what is the angle in quadrant ______ whose cosine equals \(-\frac{1}{2}\)? Answer: ______________

Example 1: Find the exact value of each expression.

a) \( \sin^{-1} (0) \) (The answer has to be an angle between _________ and ____________.)

b) \( \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \) (The answer has to be an angle from quadrant _____ or _____; which one is it? __________)

c) \( \tan^{-1} (\sqrt{3}) \) (The answer has to be an angle from quadrant ______ or ______; which one is it? __________)

**USING PROPERTIES OF INVERSE FUNCTIONS TO FIND EXACT VALUES OF COMPOSITE FUNCTIONS**

These three boxes illustrate a point we made earlier: that \( f \left( f^{-1}(x) \right) = x \) and \( f^{-1}(f(x)) = x \).

In addition, they specify the domain of each function. Notice that the domain is different depending on whether the inverse function is on the inside or the outside. If the inverse function is on the outside, the answer has to lie in the previously-mentioned quadrants.
Example 2: Find the exact value of the composite function. Notice that the inverse function is on the outside.

a) $\sin^{-1}\left[\sin\left(-\frac{\pi}{10}\right)\right]$ The angle $-\frac{\pi}{10}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (as required by the definition in the sine box on the previous page). Thus, the answer is _______________.

b) $\cos^{-1}\left[\cos\left(-\frac{5\pi}{3}\right)\right]$ The angle $-\frac{5\pi}{3}$ is not between 0 and $\pi$. What quadrant is $-\frac{5\pi}{3}$ in? ____________ So this angle actually lies in one of the allowable quadrants; we just need to rewrite it so that it is between 0 and $\pi$.

c) $\tan^{-1}\left[\tan\left(\frac{4\pi}{5}\right)\right]$ The angle $\frac{4\pi}{5}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (as required by the definition in the tangent box on the previous page). This angle is in Quadrant _______, where tangent is ____________. In what allowable quadrant does tangent have this sign? _____

d) $\tan^{-1}\left[\tan\left(-\frac{2\pi}{3}\right)\right]$ The angle $-\frac{2\pi}{3}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In fact, the angle $-\frac{2\pi}{3}$ is in Quadrant ______.

In what allowable quadrant does tangent have this sign? _______
Example 3: Find the exact value of the composite function. Notice that the inverse function is on the inside.

a) \( \cos \left[ \cos^{-1} \left( -\frac{2}{3} \right) \right] \) When the inverse cosine function is on the inside, the x-value must be between -1 and 1.

Since \(-\frac{2}{3}\) is in this interval, the answer is simply \(-\frac{2}{3}\).

b) \( \tan \left[ \tan^{-1}(-2) \right] \) When inverse tangent is on the inside, the x-value can be any real number \((-\infty < x < \infty)\).

Therefore, since -2 is a real number, then \( \tan \left[ \tan^{-1}(-2) \right] = \text{___________}. \)

c) \( \sin \left[ \sin^{-1}(-2) \right] \) When inverse sine is on the inside, the x-value must be between -1 and 1. Since -2 is NOT in this interval, we say the answer is "Not Defined".

d) \( \cos \left[ \cos^{-1}(0.5) \right] = \text{________________________} \)

e) \( \sin \left[ \sin^{-1}(\pi) \right] = \text{________________________} \)

FINDING THE INVERSE FUNCTION OF A TRIGONOMETRIC FUNCTION

We do this using the same steps we did in college algebra and precalculus. **Step 1) Replace** \( f(x) \) **with** \( y \). **Step 2) Switch** all y's and x's. **Step 3) Solve for** \( y \). **Step 4) Replace** \( y \) **with** \( f^{-1}(x) \). In Step 3, the only way to "free" the trig argument is to take the inverse trig function of both sides of the equation. For instance, \( \sin(2y) = x + 3 \) has to be solved by taking the inverse-sine of both sides: \( \sin^{-1}( \sin(2y) ) = \sin^{-1}( x + 3 ) \), freeing the argument and leaving \( 2y = \sin^{-1}( x + 3 ) \).

Then, we divide both sides by 2 to get \( y = \frac{\sin^{-1}( x + 3 )}{2} \) or \( y = \frac{1}{2} \sin^{-1}( x + 3 ) \).

Example 4: Find the inverse function and state the domain and range of \( f \) and \( f^{-1} \).

a) \( f(x) = 2\tan(x) - 3 \)

b) \( f(x) = \cos(x + 2) + 1 \)
Example 5: Find the exact solution of each equation.

a) \[ 2 \cos^{-1}(x) = \pi \]

b) \[ -6 \sin^{-1}(3x) = \pi \]

c) \[ -4 \tan^{-1}(x) = \pi \]
FIND THE EXACT VALUE OF EXPRESSIONS INVOLVING THE INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

When solving composition problems in the previous section, we were only dealing with problems like $\sin\left[\sin^{-1}(0.5)\right]$ or $\cos^{-1}\left[\cos\left(\frac{\pi}{2}\right)\right]$, and to answer these questions, we never actually had to figure out what the answer to the inside portion was. We just had to apply the properties given on page 3 of the section 8.1 notes. In this section, we now are going to have problems that mix trig functions. For instance, $\sin\left[\cos^{-1}\left(\frac{1}{2}\right)\right]$ or $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$. For these problems, we will actually have to figure out the answer to the inside portion (inside the brackets) before we can find the answer to the expression.

Example 1: Find the exact value of each expression.

a) $\sin\left[\cos^{-1}\left(\frac{1}{2}\right)\right]$ First, we must find the answer to $\cos^{-1}\left(\frac{1}{2}\right)$. So we ask ourselves: "Cosine of what angle (from quadrant _____ or _____) equals $\frac{1}{2}$?" Answer to inside: $\theta = ________$

Now we answer the outside part: $\sin \theta = \sin _____ = ____________.

b) $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$ First, the inside portion: "Sine of what angle (from quadrant _____ or _____) equals $-\frac{1}{2}$?" Answer to inside: $\theta = ________  Now we answer the outside part: $\tan \theta = \tan _____ = ____________.

KNOW THE DEFINITION OF THE INVERSE SECANT, COSECANT, AND COTANGENT FUNCTIONS

These definitions are to be expected; they are the exact same concept as the inverse sine, cosine, and tangent functions. Notice that $\sec^{-1}$ has the same range as $\cos^{-1}$ (quadrants 1 and 2), and $\csc^{-1}$ has the same range as $\sin^{-1}$ (quadrants 1 and 4). The range of $\cot^{-1}$ is quadrants 1 and 2, not including the endpoints (since cotangent has VA’s at 0 and $\pi$).
Example 2: Find the exact value of each expression.

a) \( \sec \left[ \tan^{-1} \left( \sqrt{3} \right) \right] \) First, the inside portion: "Tangent of what angle (from quadrant _____ or _____) equals \( \sqrt{3} \)?" Which quadrant do you choose? _____ Answer to inside: \( \theta = \) ___________.

Now answer the outside part: \( \sec \theta = \sec _____ = \) ___________.

b) \( \csc \left[ \tan^{-1} (-2) \right] \)

c) \( \cot \left[ \cos^{-1} \left( -\frac{\sqrt{3}}{3} \right) \right] \)

Example 3: Find the exact value of each expression.

a) \( \csc^{-1} \left( \sqrt{2} \right) \)

b) \( \sec^{-1} (-2) \)
In previous sections, we learned several trig identities, as shown below:

### Quotient Identities
\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \]

### Reciprocal Identities
\[ \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta} \quad \cot \theta = \frac{1}{\tan \theta} \]

### Pythagorean Identities
\[ \sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \]
\[ \cot^2 \theta + 1 = \csc^2 \theta \]

### Even-Odd Identities
\[ \sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta \]
\[ \csc(-\theta) = -\csc \theta \quad \sec(-\theta) = \sec \theta \quad \cot(-\theta) = -\cot \theta \]

**USE ALGEBRA TO SIMPLIFY TRIGONOMETRIC EXPRESSIONS**

For these types of problems, we will simplify expressions using the identities above and also by following any instructions given in the problem.

**Example 1:** Simplify the expression by following the indicated direction.

a) Rewrite in terms of sine and cosine functions: \( \cot \theta \cdot \sec \theta \)

b) Multiply \( \frac{\sin \theta}{1 + \cos \theta} \) by \( \frac{1 - \cos \theta}{1 - \cos \theta} \). So we are multiplying the original expression by a version of the number ____.

c) Rewrite over a common denominator: \( \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} \)
Example 1 (continued): Simplify the expression by following the indicated direction.

d) Multiply and simplify: \( \frac{(\tan \theta + 1)(\tan \theta + 1) - \sec^2 \theta}{\tan \theta} \)

e) Factor and simplify: \( \frac{\cos^2 \theta - 1}{\cos^2 \theta - \cos \theta} \)

**ESTABLISH IDENTITIES**

For the next type of problems, we will have to use some algebraic creativity to establish (i.e. prove) some identities.

Guidelines for executing these types of problems are given in the box below.

1. Only ever work with ONE side of the identity to make it equal the other side. (Never work with both sides to try to make them “meet in the middle”.)
2. Start with the side containing the more complicated expression.
3. Rewrite sums or differences of quotients as a single quotient using a Least Common Denominator.
4. Rewrite Reciprocals and Quotients in terms of sines and cosines.
5. Sometimes multiplying by a version of 1 (using a conjugate) is helpful.
6. If there is a product in the identity, then try multiplying it out and simplifying it.
7. If something is factorable, then factor it and see if any factors can cancel.

★★Any time you use an identity (QI, RI, or PI), identify it in that step of your solution! ★★
Example 2: Establish each identity.

a) \( \sec \theta \cdot \sin \theta = \tan \theta \)

b) \( \sin \theta (\cot \theta + \tan \theta) = \sec \theta \)

c) \( (\csc \theta + \cot \theta)(\csc \theta - \cot \theta) = 1 \)

d) \( (1 - \cos^2 \theta)(1 + \cot^2 \theta) = 1 \)

e) \( \csc u - \cot u = \frac{\sin u}{1 + \cos u} \)
Example 2 (continued): Establish each identity.

f) \[ 1 - \frac{\sin^2 \theta}{1 - \cos \theta} = -\cos \theta \]

g) \[ \frac{\cos \theta + 1}{\cos \theta - 1} = \frac{1 + \sec \theta}{1 - \sec \theta} \]

h) \[ \frac{\cos \nu}{1 + \sin \nu} + \frac{1 + \sin \nu}{\cos \nu} = 2 \sec \nu \]

i) \[ \frac{1 + \sin \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{1 + \sin \theta} = 4 \tan \theta \sec \theta \]

j) \[ \ln|\sec \theta + \tan \theta| + \ln|\sec \theta - \tan \theta| = 0 \]
8.4 – Sum and Difference Formulas

In this section we will learn formulas for trig functions that involve the sum or difference of two angles. These formulas are used to rewrite a trig expression when there is a sum or difference in the argument. Let’s start with the sum and difference formulas for cosine and sine:

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta
\end{align*}
\]

To help you remember these formulas, let’s think of them this way:

To expand a sum/difference within cosine: (multiply cosines together) [opposite sign] (multiply sines together)

*Just remember "CGO": For Cosine, Group like functions and use the Opposite sign*

To expand a sum/difference within sine: (multiply sine by cosine) [same sign] (multiply cosine by sine)

*Remember "SMS": For Sine, Mix the functions and use the Same sign*

*In both cases, the first trig function after the equals sign is the same as the original trig function.*

YOU NEED TO MEMORIZE THESE FORMULAS!!!

For example, let’s expand \(\cos(45^\circ + 30^\circ)\). We remember that for cosine, we are going to multiply together like functions and put the opposite sign between them. So since the original problem has a plus sign, we’ll use a minus sign in our expansion. Thus:

\[\cos(45^\circ + 30^\circ) = \ldots \times \ldots - \ldots \times \ldots\]

Since the original problem was \(\cos(45^\circ + 30^\circ)\), this could have been stated as "Find the exact value of \(\cos(75^\circ)\)."

If the problem had asked us to find \(\sin(75^\circ)\), we would rewrite it as \(\sin(45^\circ + 30^\circ)\) and remember that for Sines, we Mix sines and cosines together, starting with sine of the first angle, and we use the Same sign between the terms.

\[\sin(75^\circ) = \sin(\ldots^\circ + \ldots^\circ) = \ldots\]
Now let's look at a couple examples of angles in radians. A good rule of thumb is that if the denominator is 12, then it is a combination of $\frac{\pi}{4}$ and either $\frac{\pi}{3}$ or $\frac{\pi}{6}$, since the least common denominator of 4 and 3 (and also 4 and 6) is 12.

For help with these problems, remember that: $\frac{\pi}{4} = \frac{\pi}{12}$, $\frac{\pi}{3} = \frac{\pi}{12}$, and $\frac{\pi}{6} = \frac{\pi}{12}$.

Example 1: Find the exact value of each expression. Remember CGO and SMS!

$$\cos\left(-\frac{\pi}{12}\right) = \cos\left(-\frac{\pi}{4} - \frac{\pi}{12}\right) =$$

We can also do these problems in reverse, where we start with an expansion and condense it into a single sine or cosine function. Again, it is helpful to remember CGO and SMS (if like functions are grouped together, it is an expansion of a Cosine function, and if the functions are mixed together, then it is an expansion of a Sine function).

Example 2: Find the exact value of each expression.

a) $\sin 20^\circ \cos 80^\circ - \cos 20^\circ \sin 80^\circ$

Because the sines and cosines are mixed together, we know this is the expansion of a __________ function.

Completing the final letter of our "SMS" guide, we know that the condensed function will use the __________ sign as the expanded one, so the condensed function will be: ____________________________

Now find the exact value of this trig function.

b) $\cos \frac{5\pi}{12} \cos \frac{7\pi}{12} - \sin \frac{5\pi}{12} \sin \frac{7\pi}{12}$

Because the sines and cosines are grouped together, we know this is the expansion of a __________ function. Completing the final letter of our "CGO" guide, we know that the condensed function will use the __________ sign as the expanded one, so the condensed function will be: ____________________________

Now find the exact value of this trig function.
Now let’s learn the sum and difference formulas for tangent:

Notice that the numerator has the same sign as the original function and the denominator has the opposite sign. You do not have to memorize the sum/difference formulas for tangent because they are not used as frequently as the sine and cosine formulas.

**Example 3:** Find the exact value of each expression.

a) \( \tan \left( \frac{5\pi}{12} \right) = \tan \left( \frac{\pi}{3} + \frac{\pi}{4} \right) = \)

b) \( \frac{\tan 40^\circ - \tan 10^\circ}{1 + \tan 40^\circ \tan 10^\circ} \)

**FINDING EXACT VALUES GIVEN THE VALUES OF SOME TRIGONOMETRIC FUNCTIONS**

**Example 4:** Find the exact value of (a) \( \sin (\alpha + \beta) \), (b) \( \cos (\alpha + \beta) \), (c) \( \sin (\alpha - \beta) \), (d) \( \tan (\alpha - \beta) \) given the following information about \( \alpha \) and \( \beta \):

\( \cos \alpha = \frac{\sqrt{5}}{5} \), \( 0 < \alpha < \frac{\pi}{2} \), \( \sin \beta = -\frac{4}{5} \), \( -\frac{\pi}{2} < \beta < 0 \).

In order to use our sum and difference formulas, we need some more information. For parts (a), (b), and (c), we will need to find \( \sin (\alpha) \) and \( \cos (\beta) \) to plug into our formulas. For part (d) we will need \( \tan (\alpha) \) and \( \tan (\beta) \) also.

The problem tells us that \( \alpha \) is in Quadrant ______. So draw a triangle in that quadrant that has an adjacent side of \( \sqrt{5} \) and a hypotenuse of 5. Now use the Pythagorean Theorem to find the opposite side. Using this completed triangle, fill in the following values: \( \sin (\alpha) = \) __________ and \( \tan (\alpha) = \) ___________.
The problem also tells us that \( \beta \) is in Quadrant ______. So draw a triangle in that quadrant that has an opposite side of 4 and a hypotenuse of 5. Now use the Pythagorean Theorem to find the adjacent side side. Using this completed triangle, fill in the following values: \( \cos(\beta) = \) ___________ and \( \tan(\beta) = \) ___________.

Now we are ready to answer the questions that the original problem asked!

Rewrite all of the values from the previous page so we can have them handy as we work the problem out:

\[
\cos \alpha = \frac{\sqrt{5}}{5}, \sin \alpha = \; , \tan \alpha = \; , \cos \beta = \; , \sin \beta = -\frac{4}{5}, \tan \beta = \; 
\]

(a) \( \sin(\alpha + \beta) \)

(b) \( \cos(\alpha + \beta) \)

(c) \( \sin(\alpha - \beta) \)

(d) \( \tan(\alpha - \beta) \)
ESTABLISHING IDENTITIES USING THE SUM AND DIFFERENCE FORMULAS

Example 5: Establish each identity.

a) \( \cos (\pi - \theta) = -\cos \theta \)

b) \( \cos \left( \frac{3\pi}{2} + \theta \right) = \sin \theta \)

USING SUM AND DIFFERENCE FORMULAS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Example 6: Find the exact value of the expression \( \cos \left( \tan^{-1} \left( \frac{4}{3} \right) + \cos^{-1} \left( \frac{12}{13} \right) \right) \).

The outer function is cosine. Remember CGO: \( \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \).

Let \( \alpha = \tan^{-1} \left( \frac{4}{3} \right) \) and \( \beta = \cos^{-1} \left( \frac{12}{13} \right) \). This means that \( \tan \alpha = \frac{4}{3} \) and \( \cos \beta = \frac{12}{13} \). So we already have the value of \( \cos \beta \), but in order to use our sum formula, we still need to find \( \cos \alpha \), \( \sin \alpha \), and \( \sin \beta \). \( \tan \alpha \) and \( \cos \beta \) are both positive, which means both \( \alpha \) and \( \beta \) come from Quadrant _____.

For \( \alpha \), we draw a triangle with opposite side of 4 and adjacent side of 3, then use the Pythagorean Theorem to find that the hypotenuse is _______. So using this triangle we see that \( \sin \alpha = \frac{4}{5} \) and \( \cos \alpha = \frac{3}{5} \).

Now we move to \( \beta \), where we draw a triangle with adjacent side of 12 and hypotenuse of 13. Use the Pythagorean Theorem to find that the opposite side is _______. Using this triangle we see that \( \sin \beta = \frac{5}{13} \).

Now plug these values into the sum formula for cosine to find the answer.
Section 8.5 – Double-angle and Half-angle Formulas

DERIVING THE DOUBLE-ANGLE FORMULAS

Let's recall the Sum Formula for sine: \( \sin(\alpha + \beta) = \) __________________________

Now let's use this formula to find out what \( \sin(2\theta) \) is. We will do this by rewriting \( 2\theta \) as \( \theta + \theta \).

\[ \sin(\theta + \theta) = \] __________________________

Thus, \( \sin(2\theta) = \) __________________________ This is a very important formula – you need to memorize it!

We could similarly use the Sum Formula for cosine to find the value of \( \cos(2\theta) \), but I will spare you the details! There are 3 formulas for \( \cos(2\theta) \), and they are shown in the box below.

The only one that you need to memorize is the \( \sin(2\theta) \) formula. You do NOT need to memorize the cosine formulas. The reason for this is that the \( \sin(2\theta) \) formula is used very often in calculus, but the cosine ones are not used as often.

USE DOUBLE-ANGLE FORMULAS TO FIND EXACT VALUES

Example 1: Use the information given about the angle to find the exact value of (i) \( \sin(2\theta) \) and (ii) \( \cos(2\theta) \).

a) \( \sin\theta = -\frac{\sqrt{3}}{3} \), \( \frac{3\pi}{2} < \theta < 2\pi \). The problem tells us that \( \theta \) is in Quadrant _____, which is why \( \sin \theta \) is negative.

In this quadrant, cosine is __________________________.

Since \( \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{3} \), use the Pythagorean Theorem to find the length of the adjacent side:

Thus, the adjacent side = __________, and \( \cos \theta = \) __________.
Example 1 (continued): Use the information given about the angle to find the exact value of (i) sin(2θ) and (ii) cos(2θ).

Now that we know the values of both sin θ and cos θ, we can answer the original question.

(i) sin(2θ) =

(ii) cos(2θ) =

b) cot θ = 3, cos θ < 0

This problem is presented a little differently, because it does not tell us what quadrant θ is in. We have to figure it out using the fact that cot θ is positive and cos θ is negative (< 0). What quadrant would this be in?

Quadrant ______. In this quadrant, sine is ______________________ and cosine is ______________________.

Since cot θ = 3, that means that ___________________ = \frac{3}{1}. Use the Pythagorean Theorem to find the missing side, then find sin θ and cos θ. Lastly, answer parts (i) and (ii).
USE DOUBLE-ANGLE FORMULAS TO ESTABLISH IDENTITIES

Using the Sum Formula for Tangent, and letting both $\alpha$ and $\beta$ be equal to $\theta$, we would have the following:

$$
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
$$

$$
\tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2\tan \theta}{1 - \tan^2 \theta}.
$$

Thus, the Double-angle formula for tangent is:

$$
\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}
$$

Rearranging the double-angle formula for cosine gives us two trig formulas that are very frequently used in calculus:

$$
\begin{align*}
(\cos(2\theta) &= 1 - 2\sin^2 \theta) \rightarrow -\cos(2\theta) = -1 + 2\sin^2 \theta \\
\{\text{add 1 to both sides}\} &\rightarrow 1 - \cos(2\theta) = 2\sin^2 \theta \\
\{\text{divide by 2}\} &\rightarrow \frac{1 - \cos(2\theta)}{2} = \sin^2 \theta \quad (1)
\end{align*}

\text{AND}

\begin{align*}
(\cos(2\theta) &= 2\cos^2 \theta - 1) \\
\{\text{add 1 to both sides}\} &\rightarrow 1 + \cos(2\theta) = 2\cos^2 \theta \\
\{\text{divide by 2}\} &\rightarrow \frac{1 + \cos(2\theta)}{2} = \cos^2 \theta \quad (2)
\end{align*}

You need to memorize these two formulas for the test (and also because you will need them so often in your calculus classes!!!) Here they are as presented in your book:

$$
\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}
$$

$$
\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}
$$

You will also frequently see these formulas written with the $\frac{1}{2}$ factored out:

$$
\sin^2 \theta = \frac{1}{2} \left[ 1 - \cos(2\theta) \right]
$$

$$
\cos^2 \theta = \frac{1}{2} \left[ 1 + \cos(2\theta) \right]
$$

The remaining formula of this type is for tangent. You do NOT need to memorize this formula; it is rarely used.

$$
\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}
$$
THE HALF-ANGLE FORMULAS

In the previous three formulas, let’s let \( \theta = \frac{\alpha}{2} \). Thus, \( 2\theta = 2 \cdot \frac{\alpha}{2} = \alpha \), so each \( \cos(2\theta) \) in the formulas would become \( \cos(\alpha) \). Therefore, we would have:

\[
\begin{align*}
\sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\
\cos^2 \frac{\alpha}{2} &= \frac{1 + \cos \alpha}{2} \\
\tan^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{1 + \cos \alpha}
\end{align*}
\]

Then, take the square root of both sides of each equation. Remember that we always put a \( \pm \) symbol when we take the square root of both sides of an equation. This results in the Half-angle Formulas:

\[
\begin{align*}
\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\
\cos \frac{\alpha}{2} &= \pm \sqrt{\frac{1 + \cos \alpha}{2}} \\
\tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}
\end{align*}
\]

You do NOT have to memorize these formulas.

FINDING EXACT VALUES USING HALF-ANGLE FORMULAS

Example 2: Going back to the first example we did (on Page 2), \( \sin \theta = -\frac{\sqrt{3}}{3} \), \( \frac{3\pi}{2} < \theta < 2\pi \), let’s find two additional values: (c) \( \sin \left( \frac{\theta}{2} \right) \) and (d) \( \cos \left( \frac{\theta}{2} \right) \). Using our half-angle formulas, we see that we only need to know the value of \( \cos \theta \). When we did this example earlier, we found that \( \cos \theta = \underline{\phantom{.}} \). Now answer parts (c) and (d), remembering that sine will be \( \underline{\phantom{.}} \) and cosine will be \( \underline{\phantom{.}} \) in this quadrant.
Example 3: Use the Half-angle Formulas to find the exact value of each expression.

\[ \cos 22.5^\circ \]

b) \( \tan \left( \frac{9\pi}{4} \right) \)

c) \( \csc \left( \frac{7\pi}{8} \right) \)

Example 4: Establish each identity.

a) \( \csc(2\theta) = \frac{1}{2} \sec \theta \csc \theta \)

\[
\begin{align*}
\csc(2\theta) &\rightarrow \frac{1}{\sin(2\theta)} \\
&\rightarrow \frac{1}{2\sin \theta \cos \theta} \\
&\rightarrow \frac{1}{2 \sin \theta} \cdot \frac{1}{\cos \theta} \\
&\rightarrow \frac{1}{2} \csc \theta \sec \theta = \frac{1}{2} \sec \theta \csc \theta \ 
\end{align*}
\]

\[ \checkmark \]

b) \( \frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = \cos(2\theta) \)
Example 4 (continued): Establish each identity.

c) \( \sin^2 \theta \cos^2 \theta = \frac{1}{8} [1 - \cos(4\theta)] \)

\[
\left(\frac{1 - \cos(2\theta)}{2}\right) \cdot \left(\frac{1 + \cos(2\theta)}{2}\right) \rightarrow \{\text{FOIL}\} \rightarrow \frac{1 - \cos^2(2\theta)}{4} \rightarrow \frac{1}{4} \left[1 - \cos^2(2\theta)\right] \rightarrow \frac{1}{4} \left[1 - \frac{1 + \cos(4\theta)}{2}\right] \rightarrow \{\text{LCD}\}
\]

\[
\rightarrow \frac{1}{4} \left[\frac{2}{2} - \frac{1 + \cos(4\theta)}{2}\right] \rightarrow \{\text{distribute ‘-’ sign}\} \rightarrow \frac{1}{4} \left[\frac{1 - \cos(4\theta)}{2}\right] \rightarrow \frac{1}{4} \cdot \frac{1}{2} [1 - \cos(4\theta)] = \frac{1}{8} [1 - \cos(4\theta)] \checkmark
\]

d) \( \tan \frac{\nu}{2} = \csc \nu - \cot \nu \)

e) \( 1 - \frac{1}{2} \sin(2\theta) = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta + \cos \theta} \)  
(This one requires you to remember how to factor a sum of cubes!)

f) \( \ln|\cos \theta| = \frac{1}{2} (\ln|1 + \cos(2\theta)| - \ln 2) \)
Section 8.6 – Product-to-Sum and Sum-to-Product Formulas

EXPRESS PRODUCTS AS SUMS
The products of sines and cosines can be rewritten as sums or differences using the formulas on the right. They do NOT need to be memorized.

Example 1: Write as a sum containing only sines or cosines.
   a) \( \cos(4\theta) \cos(2\theta) \)

   b) \( \sin(3\theta) \sin(5\theta) \)

   c) \( \sin\left(\frac{3\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \)

EXPRESS SUMS AS PRODUCTS
Just as products of sines and cosines can be expressed as sums, sums and differences of sines and cosines can be expressed as products using the formulas on the right. Again, you do NOT need to memorize these formulas. They will be given to you on the test.
Example 2: Express each sum or difference as a product of sines and/or cosines.

a) \( \sin(4\theta) + \sin(2\theta) \)

b) \( \cos(5\theta) - \cos(3\theta) \)

c) \( \sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) \)

Example 3: Establish the identity

\[
\frac{\cos(\theta) + \cos(3\theta)}{2\cos(2\theta)} = \cos\theta
\]
Section 8.7 – Trigonometric Equations (I)

In this section, we are going to learn how to solve a trigonometric equation. Before we do that, let's review how to solve some basic equations from algebra.

1. **Linear Equation**
   An example is $3x + 5 = 14$. We solve this type of equation by isolating the variable. Solve this equation now.

2. **Quadratic Equation**
   An example is $2x^2 - 5x = 7$. We solve quadratic equations by setting them equal to zero and then either factoring or using the Quadratic Formula (which is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$). Solve this equation now.

3. **Basic Trigonometric Equations**
   We learned to solve some very basic trig equations back in Section 8.1 when we first learned about the inverse trigonometric functions. An example is $3\sin^{-1} x = \pi$. Just like with the linear equations, the goal is to get the variable by itself. We start by dividing both sides by 3 to get $\sin^{-1} x = \frac{\pi}{3}$. Then, we "undo" the inverse-sine function by taking the sine of both sides: $\sin(\sin^{-1} x) = \sin\left(\frac{\pi}{3}\right)$. This leaves just $x = \sin\left(\frac{\pi}{3}\right)$, so $x = \frac{\sqrt{3}}{2}$.

   Notice in this problem that we "undid" the inverse-sine function by taking the sine of both sides of the equation. Recall that $\sin(\sin^{-1} x) = x$ due to our inverse-function properties.

   Now let's look at the equation $\sin(x) = \frac{1}{2}$. If you were asked to solve for $x$, what is the first answer that pops into your head? ________________ Is that the only answer to this equation? ________________ If the problem doesn't restrict the domain, then there are ____________________________ solutions.

   You need to be able to write the solutions to these problems in a way that conveys all of the solutions. Let's do that now for this problem.
Example 1: Solve each equation on the interval $0 \leq \theta < 2\pi$.

a) $1 - \cos(\theta) = \frac{1}{2}$

b) $4\cos^2(\theta) - 3 = 0$

c) $\tan\left(\frac{\theta}{2}\right) = \sqrt{3}$

d) $5\csc(\theta) - 3 = 2$

e) $4\sin(\theta) + 3\sqrt{3} = \sqrt{3}$

f) $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$
Example 2: Solve each equation. Give a general formula for all the solutions. Then list six solutions.

a) \[ \tan(\theta) = 1 \]

b) \[ \cos(\theta) = -\frac{\sqrt{3}}{2} \]

In the next example, the argument of the trig function has a coefficient of 2. Anytime the argument of the trig function has a coefficient other than 1, you HAVE to put "+ 2\pi k" after your original answer, then solve for \( \theta \) and find all possible solutions for various values of \( k \).

c) \[ \sin(2\theta) = -1 \rightarrow 2\theta = \sin^{-1}(-1) \rightarrow 2\theta = _____ + 2\pi k \]
Section 8.8 – Trigonometric Equations (II)

In this section we will continue to solve trigonometric equations.

Example 1: Solve each equation on the interval $0 \leq \theta < 2\pi$.

a) $2\cos^2 \theta + \cos \theta - 1 = 0$

This equation is "quadratic in form", meaning that the exponent of the first term is double the exponent of the middle term, and the last term is a constant. Just like a regular quadratic equation, we will solve this by factoring (if possible) or using the Quadratic Formula. Make sure you find all of the solutions between 0 and $2\pi$.

b) $\cos^2 \theta - \sin^2 \theta + \sin \theta = 0$

If the equation has multiple terms, as this equation does, then we generally do not want a mixture of trig functions. So we want to try to get this equation entirely in terms of sines. What formula or identity can we use to turn the cosine-squared term into something using sines? ________________________ Rewriting the equation gives: ________________________

______________________________.
Example 2: Solve each equation on the interval \(0 \leq \theta < 2\pi\).

a) \(\cos(2\theta) = 2 - 2\sin^2 \theta\)

This problem is good for us to do because it shows that despite our best efforts, sometimes things just go down the crapper... 😊

\[
\cos(2\theta) = 2 - 2\sin^2 \theta \\
1 - 2\sin^2 \theta = 2 - 2\sin^2 \theta \quad \text{(Applied the Double-Angle Formula for Cosine on the Left)} \\
1 = 2 \quad \text{(Added } 2\sin^2 \theta \text{ to both sides)}
\]

Since the variable \(\theta\) cancelled out and we are left with a FALSE statement, there is NO SOLUTION to this equation.

b) \(\sin(2\theta) = \cos \theta\). Start by using the Double-angle Formula.

\[
2\sin \theta \cos \theta = \cos \theta - \cos \theta \rightarrow 2\sin \theta \cos \theta - \cos \theta = 0 \quad \text{Factor out } \cos \theta \rightarrow \cos \theta \cdot (2\sin \theta - 1) = 0
\]

Because we have a product that equals ZERO, we can apply the Zero Product Principle (the same thing we do when we solve a problem by factoring – we set each factor equal to zero and solve for the variable).

➢ **First factor:** \(\cos(\theta) = 0 \text{ when } \theta = \) \_\_\_\_\_\_\_\_\_\_\_\_ \text{ or } \_\_\_\_\_\_\_\_\_\_. \text{ The next place cosine would equal zero is at } \_\_\_\_\_\_, \text{ which is outside of our domain, so those are the only two answers from that factor.}

➢ **Second factor:** \(2\sin \theta - 1 = 0 \rightarrow \frac{\sin \theta = \frac{1}{2}}{\text{Add } 1} \rightarrow 2\sin \theta = 1 \rightarrow \frac{\sin \theta = \frac{1}{2}}{+ 2}\)

For what Q1 angle does sine equal \(\frac{1}{2}\)? \_\_\_\_\_\_

What other quadrant has positive sine? \_\_\_\_\_\_\_\_\_

What is the angle in that quadrant that has a sine value of \(\frac{1}{2}\)? \_\_\_\_\_\_\_\_\_

So the solutions to this problem are:
Example 3: Solve the equation on the interval \(0 \leq \theta < 2\pi\).

\[
\cos \theta + \sin \theta = 0
\]

Start by subtracting \(\cos \theta\) from both sides. This gives us \(\sin \theta = -\cos \theta\).

Can you think of a reference angle (in Quadrant 1) where sine and cosine have the same value? 

Now, in what quadrant would we find positive sine and negative cosine? 

What is the angle in that quadrant with the previously-mentioned reference angle? 

Now let's subtract \(\sin \theta\) from both sides. This gives us \(\cos \theta = -\sin \theta\).

In what quadrant would we find positive cosine and negative sine? 

What is the angle in that quadrant with the previously-mentioned reference angle? 

Example 4: Solve the equation on the interval \(0 \leq \theta < 2\pi\).

\[
\sin(4\theta) - \sin(6\theta) = 0
\]

Sum-to-Product Formula

\[
\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(5\theta) = 0
\]

Because we have a product that equals zero, we can apply the Zero Product Principle.

- **First factor:** \(\sin(\theta) = 0\) when \(\theta = \) \(\) or \(\).
  
  We know that the next place sine would equal zero is at \(2\pi\), which is outside of our domain, so those are the only two answers from that factor. That's the easy one!

- **Second factor:** \(\cos (5\theta) = 0\). Cosine equals zero at \(\) and \(\).

Now the fun part... Because the argument is \(5\theta\) instead of just \(\theta\), we have to do the "\(+ 2\pi k\)" trick to find all the solutions.

So we have \(5\theta = \frac{\pi}{2} + 2\pi k\) and thus \(\theta = \frac{\pi}{10} + \frac{2\pi k}{5}\) \(\text{LCD is } 10\) so multiply by \(\frac{2}{2}\),

\[
\Rightarrow \quad \frac{\pi + 4\pi k}{10} \quad \text{1}
\]

And we also have \(5\theta = \frac{3\pi}{2} + 2\pi k\) and \(\theta = \frac{3\pi}{10} + \frac{2\pi k}{5}\) \(\text{LCD is } 10\) so multiply by \(\frac{2}{2}\),

\[
\Rightarrow \quad \frac{3\pi + 4\pi k}{10} \quad \text{2}
\]
Example 4 (continued):

Now make a table of possible solutions from the second factor and check to see which ones are in the domain

\(0 \leq \theta < 2\pi\):

<table>
<thead>
<tr>
<th>(k)</th>
<th>(1) (\theta = \pi + 4\pi k/10)</th>
<th>(2) (\theta = 3\pi + 4\pi k/10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k = 0)</td>
<td>(\theta = \pi/10 \checkmark)</td>
<td>(\theta = 3\pi/10 \checkmark)</td>
</tr>
<tr>
<td>(k = 1)</td>
<td>(\theta = (\pi + 4\pi)/10 = \frac{5\pi}{10} = \frac{\pi}{2} \checkmark)</td>
<td>(\theta = 3\pi + 4\pi/10 = \frac{7\pi}{10} \checkmark)</td>
</tr>
<tr>
<td>(k = 2)</td>
<td>(\theta = (\pi + 8\pi)/10 = \frac{9\pi}{10} \checkmark)</td>
<td>(\theta = 3\pi + 8\pi/10 = \frac{11\pi}{10} \checkmark)</td>
</tr>
<tr>
<td>(k = 3)</td>
<td>(\theta = (\pi + 12\pi)/10 = \frac{13\pi}{10} \checkmark)</td>
<td>(\theta = 3\pi + 12\pi/10 = \frac{15\pi}{10} = \frac{3\pi}{2} \checkmark)</td>
</tr>
<tr>
<td>(k = 4)</td>
<td>(\theta = (\pi + 16\pi)/10 = \frac{17\pi}{10} \checkmark)</td>
<td>(\theta = 3\pi + 16\pi/10 = \frac{19\pi}{10} \checkmark)</td>
</tr>
<tr>
<td>(k = 5)</td>
<td>(\theta = (\pi + 20\pi)/10 = \frac{21\pi}{10} \times)</td>
<td>(\theta = 3\pi + 20\pi/10 = \frac{23\pi}{10} \times)</td>
</tr>
</tbody>
</table>

Note that \(\frac{21\pi}{10}\) and \(\frac{23\pi}{10}\) are not in our domain because they are greater than \(\frac{20\pi}{10}\) (which is \(2\pi\)).

Also, notice that to get from one answer to the next in each column, all you do is add \(4\pi/10\) to the previous answer (this is because of the "\(+ 4\pi k\)" in the numerator of the general solution).

So all of the solutions for this problem are:
Example 5: Solve the equation on the interval $0 \leq \theta < 2\pi$.

$$4(1 + \sin \theta) = \cos^2 \theta \quad (\text{Hint: Distribute on the left and use the Pythagorean Identity on the right})$$

Example 6: Solve the equation on the interval $0 \leq \theta < 2\pi$.

$$\cos(2\theta) + 5\cos \theta + 3 = 0 \quad (\text{Hint: Start with the Double-Angle Formula } \cos(2\theta) = 2\cos^2 \theta - 1.)$$
Example 7: Solve the equation on the interval $0 \leq \theta < 2\pi$.

$$\sec \theta = \tan \theta + \cot \theta$$

This is the first equation we have been presented with that contains trig functions other than sine and cosine. The domain of both sine and cosine is **all real numbers**, so we never had to worry if our answer was in the domain of the function. But the domain of secant **excludes** ____________________, the domain of tangent **excludes** ____________________, and the domain of cotangent **excludes** ____________________, so if we get any of those values as our answer, then they are not in the domain, so they can not be a solution of the equation.

\[
\begin{align*}
\sec \theta &= \tan \theta + \cot \theta & \text{rewrite as sines/cosines} \\
\frac{1}{\cos \theta} &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} & \text{Multiply each term by } \sin \theta \cos \theta \\
\sin \theta &= \sin^2 \theta + \cos^2 \theta & \text{Pythagorean identity} \\
\sin \theta &= 1
\end{align*}
\]

So we ask ourselves, sine of what angle equals 1? The answer: ____________.

Example 8: Find the real zeros of $f(x) = \cos(2x) + \sin^2 x$ on the interval $0 \leq \theta < 2\pi$. Remember that "real zeros" are the $x$-intercepts of the function, and we have $x$-intercepts when the function equals _________. So just set the function equal to _______ and solve for $x$. 

---

8.8

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For the duration of this class, we will label the sides and angles of triangles so that the angles are in CAPITAL letters and the sides are in lowercase letters, and each side will bear the same letter as the angle opposite it. Notice in the figure to the right that side "a" is opposite angle "A" and side "b" is opposite angle "B". And, as with any right triangle, $a^2 + b^2 = c^2$, and since the sum of all angles must equal 180°, and the right angle is 90°, then the other two angles must add up to 90° ($A + B = 90°$).

Example 1: Use the given information to solve the triangle.

a) $b = 4$, $B = 10°$; find $a$, $c$, and $A$.

Since $B = 10°$, then $A = 90° - 10° = 80° \rightarrow A = 80°$

$$\sin B = \frac{b}{c} \rightarrow \sin(10°) = \frac{4}{c} \rightarrow c = \frac{4}{\sin(10°)} \rightarrow c \approx 23.04$$

$$\tan B = \frac{b}{a} \rightarrow \tan(10°) = \frac{4}{a} \rightarrow a = \frac{4}{\tan(10°)} \rightarrow a \approx 22.69$$

We could have used any of the trig functions with angle $A$ or $B$ to find the values of $a$ and $c$. For instance, we could have said $\cos A = \frac{b}{c}$ to find $c$ or $\sin A = \frac{a}{c}$, so the trig functions and angles I chose to use were arbitrary.

b) $a = 2$, $b = 8$; find $c$, $A$, and $B$. 
APPLICATIONS IN NAVIGATION AND SURVEYING

In navigation and surveying, the "direction" or "bearing" is the acute angle between any point and the north-south line. The angles are labeled as North-or-South (degree measurement) East-or-West. For example, P4 in the figure to the right would be labeled as _____________________________. Remember that, unlike a typical coordinate system, you are measuring the angle from the vertical, not the horizontal!

Example 2: #30) Finding the Bearing of a Ship

A ship leaves the port of Miami with a bearing of S80°E and a speed of 15 knots. After 1 hour, the ship turns 90° toward the south. After 2 hours, maintaining the same speed, what is the bearing to the ship from port?
Example 3: #32) A surveyor in Chicago is surveying the Sears Tower, which is topped by a tall antenna. From the ground, he makes the following measurements:

1) The angle of elevation from his position to the top of the building is 34°.
2) The distance from his position to the top of the building is 2593 feet.
3) The distance from his position to the top of the antenna is 2743 feet.

Using this information, answer the following questions.

a) How far away from the base of the building is the surveyor located?

b) How tall is the building?

c) What is the angle of elevation from the surveyor to the top of the antenna?

d) How tall is the antenna?
Section 9.2 – The Law of Sines

Up until now we have dealt only with right triangles. The Pythagorean Theorem and our Soh-Cah-Toa Rules can ONLY be used for RIGHT triangles. But obviously there are other types of triangles (called oblique triangles) that do not contain a right angle. An oblique triangle can have all three angles acute (less than 90°) or one obtuse angle (between 90° and 180°) and two acute angles, as shown in the figure to the right.

Looking at the triangle to the left, you can see that there are 6 pieces of information to find – three angles and three side lengths. In this section we will use the Law of Sines to "solve" an oblique triangle, which means that we will be given 3 of the pieces of information and use it to figure out the other 3 pieces of missing information.

The Law of Sines is actually three separate equalities that can be written compactly as shown below.

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1)
\]

You will never use all three parts of the Law of Sines at once. For instance, you can use the "A" fraction with the "B" fraction, the "A" fraction with the "C" fraction, or the "B" fraction with the "C" fraction. Also, you must have 3 of the 4 pieces of information and then solve the equation to find the 1 missing piece of information.

For instance, if you are given the angle \( A = 80^\circ \), the side \( a = 2 \), and the angle \( C = 60^\circ \), then you would set up the equation:

\[
\frac{\sin A}{a} = \frac{\sin C}{c} \rightarrow \frac{\sin80^\circ}{2} = \frac{\sin60^\circ}{c}
\]

then solve for "c".

We will use the Law of Sines to solve oblique triangles of Case 1, where we know the side between two known angles (type ASA) or two angles and a different side (type SAA), and also Case 2, where we know the lengths of two sides and the angle opposite one of the sides (type SSA). In the figure below, the BLUE labels represent KNOWN quantities and the RED labels represent those values that we have to solve for.
CASE 1: SOLVING SAA OR ASA TRIANGLES

Case 1 problems are very straightforward. Simply label a triangle, remembering that sides are opposite of the angles of the same name (except that the angles are in CAPITAL letters and the sides are in lowercase letters). Then create ratios using the Law of Sines, where three of the four quantities are known, and solve for the unknown quantity.

Example 1 (ASA): #20) \( A = 70^\circ, B = 60^\circ, c = 4 \).

Example 2 (SAA): #18) \( A = 50^\circ, C = 20^\circ, a = 3 \)
CASE 2: SOLVING SSA TRIANGLES

Case 2 (SSA) triangles are so not straightforward that they are actually known as the ambiguous case. This is because the known information may result in no triangle, one right triangle, one oblique triangle, or two triangles! Because sine is positive in Quadrants 1 and 2, you must be careful when using the Law of Sines to make sure that you find the angle in Q1 AND the angle in Q2 that makes the equation true. For instance, if we had \( \sin A = \frac{1}{2} \), your first instinct would be to say that \( A = 30^\circ \). But \( A \) could also equal the angle in Quadrant 2 whose reference angle is \( 30^\circ \): that angle is _________.

We can always find the Q2 angle by doing: \( 180^\circ \) – (the Q1 angle). If the sum of the Quadrant 2 angle you find plus the angle given in the problem is less than \( 180^\circ \), then there are two triangles. If the sum of the Quadrant 2 angle and the given angle is greater than \( 180^\circ \), then there is only one triangle.

We will do one example of each type.

Example 3 (one triangle): #26) \( b = 4, c = 3, B = 40^\circ \).

\[
\sin C = \frac{\sin 40^\circ}{3} \implies \sin C = \frac{3 \cdot \sin 40^\circ}{4} \implies C = \sin^{-1}(0.482) \approx 28.8^\circ \quad \text{(Quadrant 1 angle)}
\]

\[
C_2 = 180^\circ - 28.8^\circ \approx 151.2^\circ \quad \text{(Quadrant 2 angle)}
\]

This is the point at which we determine if we are dealing with one or two triangles. The only way we could have 2 triangles is if the sum of the given angle plus the found angle is less than \( 180^\circ \). So let’s see how many triangles we get in this example: \( B + C_2 = \)

Now let's proceed to find the remaining missing information for this triangle (we are missing the measurement for angle A and the length of side a).

\[
A = 180^\circ - 40^\circ - 28.8^\circ \approx 111.2^\circ
\]

\[
\sin 40^\circ = \frac{\sin 111.2^\circ}{a} \implies a = \frac{4 \cdot \sin 111.2^\circ}{\sin 40^\circ} \approx \text{______}
\]
Example 4 (two triangles): #30) \(b = 2, \ c = 3, \ B = 40^\circ\).

\[
\sin C = \frac{\sin 40^\circ}{2} \rightarrow \sin C = \frac{3 \cdot \sin 40^\circ}{2} \rightarrow \sin C \approx 0.964 \rightarrow C = \sin^{-1}(0.964)
\]

\(C_1 \approx 74.6^\circ\) (Quadrant 1 angle)

\(C_2 = 180^\circ - 74.6^\circ \rightarrow C_2 \approx 105.4^\circ\) (Quadrant 2 angle)

Let's determine if we are dealing with one or two triangles.

\[B + C_2 = 40^\circ + 105.4^\circ = 145.4^\circ < 180^\circ.\]

Since the sum is less than 180°, we have ________________,
a drawing of which is shown in the figure to the right.

Let's start by labeling everything we know about the first (outer) triangle (the one with angle \(C_1 \approx 74.6^\circ\)).

For this triangle, we are missing the measurement for angle \(A_1\) and the length of side \(a_1\).

\[A_1 = 180^\circ - B - C_1 = 180^\circ - 40^\circ - 74.6^\circ \rightarrow A_1 \approx \text{______}\]

The bottom side is \(a_1\):

\[\frac{\sin B}{b} = \frac{\sin A_1}{a_1}\]

Now let's label the second (inner) triangle (the one with angle \(C_2 \approx 105.4^\circ\)).

\[A_2 = 180^\circ - B - C_2 = \]

The bottom side of the inner triangle is \(a_2\):

\[\frac{\sin B}{b} = \frac{\sin A_2}{a_2}\]

In summary:

Triangle 1 has side lengths \(a_1 = \text{______}, \ b = \text{______}, \ c = \text{______}, \ & \) angles \(A_1 = \text{______}, \ B = \text{______}, \ C_1 = \text{______}.

Triangle 2 has side lengths \(a_2 = \text{______}, \ b = \text{______}, \ c = \text{______}, \ & \) angles \(A_2 = \text{______}, \ B = \text{______}, \ C_2 = \text{______}.

Notice that the 3 pieces of information that were given in the problem must be the same for both triangles!
Example 5 (no triangle): #32) $a = 3, b = 7, A = 70^\circ$

\[
\frac{\sin B}{b} = \frac{\sin A}{a} \rightarrow \frac{\sin B}{7} = \frac{\sin 70^\circ}{3} \rightarrow \sin B = \frac{7 \cdot \sin 70^\circ}{3} \rightarrow \sin B \approx 2.193
\]

There is no angle for which $\sin B = 2.193$ (since we know the sine function bounces between _____ and _____).

Thus, there is no triangle that can have the given measurements.

Notice in the triangle on the left that there is no way you can position side $a$ that it will ever touch side $c$ to form a triangle.

### SOLVING APPLIED PROBLEMS USING THE LAW OF SINES

Example 6: #38) To find the distance from the house at $A$ to the house at $B$, a surveyor measures $\angle BAC$ to be $35^\circ$ and then walks off a distance of 100 feet to $C$ and measures $\angle ACB$ to be $50^\circ$. What is the distance from $A$ to $B$?

In order to use the Law of Sines, we need to know the measurement of the angle that is opposite the known side length. So we find $\angle ABC$ to be $180^\circ - 35^\circ - 50^\circ = \underline{95^\circ}$. Now we can apply the Law of Sines to find the missing length (which is the side opposite the $50^\circ$ angle).
Example 7: #42) The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado. Sightings to the same point at water level directly under the bridge are taken from each side of the 880-foot-long bridge, as indicated in the figure. How high is the bridge?

This is an _____________ triangle. Once again, we need to start by finding the measurement of the angle opposite of the known side.

Now use the Law of Sines to find the length of one of the other sides of the triangle.

Lastly, use the "Soh Cah Toa" concept to find the height of the bridge, $h$. ("Soh Cah Toa" is only valid for right triangles. Notice how $h$ splits the outer oblique triangle into two right triangles, so we can use it here!)
Section 9.3 – The Law of Cosines

In this section we will use the Law of Cosines to solve Case 3 (SAS) and Case 4 (SSS) triangles.

**Case 3:** Two sides and the included angle are known (SAS).

**Case 4:** Three sides are known (SSS).

<table>
<thead>
<tr>
<th>Law of Cosines</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a triangle with sides $a$, $b$, $c$ and opposite angles $A$, $B$, $C$, respectively,</td>
</tr>
<tr>
<td>$c^2 = a^2 + b^2 - 2ab \cos C$</td>
</tr>
<tr>
<td>$b^2 = a^2 + c^2 - 2ac \cos B$</td>
</tr>
<tr>
<td>$a^2 = b^2 + c^2 - 2bc \cos A$</td>
</tr>
</tbody>
</table>

It is really unnecessary to write out all three of these equations, because they all display the same concept:

"The square of one side of a triangle equals the sum of the squares of the other two sides minus 2 times their product times the cosine of the angle between them."

The Law of Cosines, as shown above, makes it easy to find a missing side (remember that the lower-case letters are side lengths and the CAPITAL letters are angle measurements). But often we need to use the Law of Cosines to find angle measurements. So let's solve these equations for $A$, $B$, and $C$. We'll start by using the first equation in the box above.

\[ c^2 = a^2 + b^2 - 2ab \cos C \]

Subtract $a^2$ and $b^2$ to get

\[ c^2 - a^2 - b^2 = -2ab \cos C \]

Add $-2ab$ to both sides

\[ c^2 - a^2 - b^2 - 2ab = -2ab \cos C \]

Now, factoring out a -1 in the numerator gives:

\[ -1 \cdot \left(-c^2 + a^2 + b^2\right) = -2ab \cos C \]

Cancelling the negative signs gives:

\[ \frac{-c^2 + a^2 + b^2}{2ab} = \cos C \]

Rewriting the numerator gives:

\[ \frac{a^2 + b^2 - c^2}{2ab} = \cos C \]

Then taking the inverse-cosine of both sides gives:

\[ C = \cos^{-1} \left( \frac{a^2 + b^2 - c^2}{2ab} \right) \]

Applying the same concept to the other two angles, we get the following formulas:

\[ A = \cos^{-1} \left( \frac{b^2 + c^2 - a^2}{2bc} \right), \quad B = \cos^{-1} \left( \frac{a^2 + c^2 - b^2}{2ac} \right), \quad \text{and} \quad C = \cos^{-1} \left( \frac{a^2 + b^2 - c^2}{2ab} \right) \]
SOLVE CASE 3 (SAS) TRIANGLES

Example 1: Solve each triangle.

a) \( a = 2, \ c = 1, \ B = 10^\circ \)

We will start by finding the length of side \( b \).

\[
b^2 = a^2 + c^2 - 2ac \cdot \cos B \rightarrow b^2 = 2^2 + 1^2 - 2(2)(1) \cdot \cos 10^\circ
\]

\[
\rightarrow b^2 = 1.06077 \rightarrow b \approx 1.03
\]

Now we can use the Law of Sines (from Section 9.2) OR the Law of Cosines to find the two remaining angles, \( A \) and \( C \). However, because you have to find both the Q1 and Q2 angles if you use the Law of Sines, and then try to determine which one is the correct angle, it’s easier to just use the Law of Cosines to find another angle. We will never find two answers if we use the Law of Cosines.

Since we already have angle \( B \), then we need to find either angle \( A \) or angle \( C \) using the equations from the bottom of page 1:

\[
A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}\left(\frac{1.03^2 + 1^2 - 2^2}{2 \cdot 1.03 \cdot 1}\right) = \cos^{-1}\left(\frac{-1.9391}{2.06}\right) = 160.3^\circ
\]

or:

\[
C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{2^2 + 1.03^2 - 1^2}{2 \cdot 2 \cdot 1.03}\right) = \cos^{-1}\left(\frac{4.0609}{4.12}\right) = 9.7^\circ
\]

You do NOT need to use the Law of Cosines to find both angles. Of course, you can (both angles \( A \) and \( C \) are correct). But it is much easier to just use Law of Cosines to find ONE of the angles (let’s say, angle \( A \)), then do \( 180^\circ - B - A \) to find angle \( C \).

\[
180^\circ - \frac{10^\circ}{B} - \frac{160.3^\circ}{A} = \frac{9.7^\circ}{C} \quad \checkmark
\]
Example 1 (continued): Solve each triangle.

b) $a = 6, b = 4, C = 60^\circ$

Start by finding the length of side $c$.

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C \rightarrow$$

Now find the measurements for angles $A$ and $B$.

c) $b = 4, c = 1, A = 120^\circ$

Start by finding the length of side $a$.

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A \rightarrow$$
SOLVE CASE 4 (SSS) TRIANGLES

Example 2: Solve each triangle.

a) \(a = 4, b = 5, c = 3\)

Since we are going to be solving for angles in this problem, use the angle formulas at the bottom of page 1:

\[
A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) = \cos^{-1}\left(\frac{5^2 + 3^2 - 4^2}{2 \cdot 5 \cdot 3}\right) = \cos^{-1}\left(\frac{18}{30}\right) \approx 53.1^\circ
\]

\[
B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) = \cos^{-1}\left(\frac{4^2 + 3^2 - 5^2}{2 \cdot 4 \cdot 3}\right) = \cos^{-1}(0) = 90^\circ \text{ or } \\
\]

(but the latter isn’t possible — why? )

At this point, we could finish the problem by saying that \(C = 180^\circ - A - B = 180^\circ - 53.1^\circ - 90^\circ = 36.9^\circ\).

Let’s see if this is what we get using the Law of Cosines formula:

\[
C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) = \cos^{-1}\left(\frac{4^2 + 5^2 - 3^2}{2 \cdot 4 \cdot 5}\right) = \cos^{-1}\left(\frac{32}{40}\right) \approx 36.9^\circ \checkmark
\]

So our final answer is: \(A = 53.1^\circ, B = 90^\circ, C = 36.9^\circ\) This actually is a ____________ triangle!

b) \(a = 9, b = 7, c = 10\)
Example 2 (continued): Solve each triangle.

c) \(a = 3, b = 3, c = 2\)

SOLVING APPLICATION PROBLEMS USING THE LAW OF COSINES

Example 3: #34) An airplane flies due north from Ft. Myers to Sarasota, a distance of 150 miles, and then turns through an angle of 50° and flies to Orlando, a distance of 100 miles. (a) How far is it directly from Ft. Myers to Orlando? (b) What bearing should the pilot use to fly directly from Ft. Myers to Orlando?
Example 4: #36) In attempting to fly from Chicago to Louisville, a distance of 330 miles, a pilot inadvertently took a course that was $10^\circ$ in error. (a) If the aircraft maintains an average speed of 220 miles per hour and if the error in direction is discovered after 15 minutes, through what angle should the pilot turn to head toward Louisville? (b) What new average speed should the pilot maintain so that the total time of the trip is 90 minutes?

Example 5: #38) According to Little League baseball official regulations, the diamond is a square 60 feet on a side. The pitching rubber is located 46 feet from home plate on a line joining home plate and second base. (a) How far is it from the pitching rubber to first base? (b) How far is it from the pitching rubber to second base? (c) If a pitcher faces home plate, through what angle does he need to turn to face first base?
Section 9.4 – Area of a Triangle

You probably remember learning in some previous math class that the formula for the area of a triangle is \( A = \frac{1}{2}bh \) (area equals one-half base times height). But since we always have one angle of a triangle labeled with the letter "A", we use the letter "K" instead. So, assuming you are given (or can easily find) the height, \( h \), then the formula to use is \( K = \frac{1}{2}bh \). However, we often are given triangles where the height (also called altitude) is not known. In this case, you determine if you have an SAS or SSS triangle, and use one of the formulas below.

For an SAS triangle like the one shown below, \( \sin C = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{a} \), so multiplying both sides by \( a \), we get \( a \sin C = h \). Thus, in the formula \( K = \frac{1}{2}bh \), we can replace the "h" with "\( a \sin C \)" and get \( K = \frac{1}{2}b \cdot a \sin C \), or \( K = \frac{1}{2}ab \sin C \).

In words: the area of a triangle equals one-half the product of two of its sides times the sine of the angle between them.

Example 1: Find the area of the triangle. Round answers to two decimal places.

a) \( a = 2, c = 1, B = 10^\circ \). In this case, we know the lengths of two sides, \( a \) and \( c \), and the measurement of the angle between them, \( B \). So our formula will be \( K = \frac{1}{2}ac \sin B \). \( K = \frac{1}{2} \cdot (2) \cdot (1) \sin 10^\circ \approx 0.17 \)

b) \( b = 4, c = 1, A = 120^\circ \). \( K = \) ________________________________

c) \( a = 6, b = 4, C = 60^\circ \).
\( K = \) ________________________________

When you are given all three side lengths but none of the angle measurements, then you have an SSS triangle. For these types of triangles we use Heron's Formula to find the area. To use the formula, you first calculate the value of \( s \), then you plug \( a, b, c, \) and \( s \) into the square root as shown.

**Heron’s Formula**

The area \( K \) of a triangle with sides \( a, b, \) and \( c \) is

\[
K = \sqrt{s(s - a)(s - b)(s - c)}
\]

where \( s = \frac{1}{2}(a + b + c) \).
Example 2: Find the area of the triangle. Round answers to two decimal places.

a) $a = 4, b = 5, c = 3$.

First we find $s$. 
$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(4 + 5 + 3) = \frac{1}{2}(12) = 6$$

Now plug into the square root:
$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{6(6-4)(6-5)(6-3)} = \sqrt{6(2)(1)(3)} = \sqrt{36} = 6$$

So the area of this triangle is 6.

b) $a = 3, b = 3, c = 2$.

c) $a = 4, b = 3, c = 6$.

Example 3: #42) The Bermuda Triangle is roughly defined by Hamilton, Bermuda; San Juan, Puerto Rico; and Fort Lauderdale, Florida. The distance from Hamilton to Ft. Lauderdale is approximately 1028 miles, from Ft. Lauderdale to San Juan is approximately 1046 miles, and from San Juan to Hamilton is approximately 965 miles. Ignoring the curvature of Earth, approximate the area of the Bermuda Triangle.
Section 10.1 – Polar Coordinates

Up until now, we have always graphed using the rectangular coordinate system (also called the Cartesian coordinate system). In this section we will learn about another system, called polar coordinates. In polar coordinates, we choose a point, called the pole, that coincides with the "origin" in rectangular coordinates. The right-facing horizontal ray that originates at the pole is called the polar axis. (The polar axis coincides with the positive portion of the x-axis in the rectangular coordinate system.)

In the rectangular system, a point is denoted by (x, y) coordinates. But in the polar system, a point P is denoted by (r, θ) coordinates, where r is the distance from the point to the pole (if r is positive... we'll discuss negative r in a moment). If we draw a ray from the pole through P, then θ is the angle (in degrees or radians) between the polar axis and that ray.

In the graphing area to the right, the polar axis is shown as the bold ray. Each circle is centered at the pole (denoted by the O), with its radius marked along the polar axis. Give the coordinates (r, θ) of each point on the graph, assuming θ is in radians and r is positive. It may help to label the following angles: \(\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}\).

P₁: ______________  P₂: ______________  P₃: ______________

P₄: ______________  Now plot the point P₅: \(\left(1, \frac{3\pi}{2}\right)\).

As we've just practiced, to plot a point with a positive r-value, you travel θ from the polar axis and then plot the point a distance of r from the pole. But if the r-value is negative, you travel to θ, then plot the point a distance of |r| in the opposite direction of θ.

For instance, to plot the point \(\left(-3, \frac{2\pi}{3}\right)\), you would locate the angle \(\frac{2\pi}{3}\), then move 3 away from the pole in the opposite direction, as shown in the figure to the left.

The point \(\left(-2, \frac{\pi}{4}\right)\) is shown in the figure on the right.
There are actually infinitely many ways to label a point in the polar coordinate system. For example, the point \( \left( 2, \frac{\pi}{4} \right) \) shown in the figure on the left could also be labeled as \( \left( 2, \frac{9\pi}{4} \right) \) since traveling one complete rotation from \( \frac{\pi}{4} \) takes us to \( \frac{\pi}{4} + 2\pi = \frac{9\pi}{4} \).

If we traveled to \( \theta = \frac{5\pi}{4} \) then moved a distance of 2 in the opposite direction, we would arrive at this same point, and we would label it \( \left( -2, \frac{5\pi}{4} \right) \).

Or, if we moved in the clockwise direction from the polar axis, then \( \theta \) would be negative, and specifically, this angle would be \( \theta = -\frac{7\pi}{4} \), so the point would be labeled \( \left( 2, -\frac{7\pi}{4} \right) \).

This example shows four ways to label this point, but as mentioned earlier, there are infinitely many ways to label this point in the polar coordinate system.

**Example 1:** Plot each point in polar coordinates, then find other polar coordinates of the point for which:

(a) \( r > 0, \ -2\pi \leq \theta < 0 \) 
(b) \( r < 0, \ 0 \leq \theta \leq 2\pi \) 
(c) \( r > 0, \ 2\pi \leq \theta < 4\pi \)
CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

It is very simple to convert from polar to rectangular coordinates. You just need to **memorize** two simple formulas:

\[
x = r \cos \theta \\
y = r \sin \theta
\]

**Example 2**: Find the rectangular coordinates of each point.

#42) \(( -3, 4\pi )\)

#46) \( \left( -2, \frac{2\pi}{3} \right) \)

#48) \( \left( -3, -\frac{3\pi}{4} \right) \)

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

It is a bit more work to convert from rectangular to polar coordinates, but it is still pretty easy to do. Sometimes you can figure it out just by "eyeballing" the point on a graph.

Here is an example of a point that you can figure out just by looking at where it is on a graph: \((0, 2)\). Remember, this is a point in **rectangular coordinates**, so \(x = 0\) and \(y = 2\). Plot this point in rectangular coordinates. Look at the graph and determine the radius (distance from the pole to the point) and the angle \(\theta\) made with the polar axis.

If we considered a positive \(\theta\), what would the coordinates be? ________________

If we instead considered a negative \(\theta\), what would the coordinates be? ________________
Example 3: Convert the point to polar coordinates (the $45^\circ$-$45^\circ$-$90^\circ$ triangle will be helpful for this problem).

$(2, -2)$

Other times you will need to use these formulas to convert from rectangular to polar coordinates:

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad \text{if} \ x \neq 0$$

Example 4: Convert the point $(-2, -2\sqrt{3})$ to polar coordinates.

First we must determine which Quadrant this point lies in. $x = -2$ and $y = -2\sqrt{3}$, so the point is in Quadrant ______.

Now we find $r$: $r^2 = (-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$. Thus, $r = \sqrt{16} = 4$.

Then we find $\theta$: $\tan \theta = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \Rightarrow \theta = \tan^{-1}(\sqrt{3}) = \underline{\quad}$ (in Q1) or $\underline{\quad}$ (in Q3).

Since we already decided that this point is in Q____, the polar coordinates for this point are: ______________.

Example 5: Convert the point $(\sqrt{3}, 1)$ to polar coordinates.
TRANSFORM EQUATIONS FROM POLAR TO RECTANGULAR FORM AND VICE-VERSA

Two common techniques for transforming an equation from polar form to rectangular form are

1. Multiplying both sides of the equation by \( r \)
2. Squaring both sides of the equation.

In addition to the formula \( r^2 = x^2 + y^2 \), it is also often helpful to use its square root: \( r = \sqrt{x^2 + y^2} \).

Example 6: Write the equation using rectangular coordinates \((x, y)\).

#78) \( r = \sin \theta - \cos \theta \)

We will start by using the first technique (multiplying both sides by \( r \)).

\[
\begin{align*}
   r \cdot r &= r \cdot (\sin \theta - \cos \theta) \\
   r^2 &= r \sin \theta - r \cos \theta \\
   Apply \ Definitions \quad &\Rightarrow \quad x^2 + y^2 = y - x
\end{align*}
\]

#76) \( r = \sin \theta + 1 \)

#80) \( r = 4 \)

To transform an equation from rectangular to polar form, you usually just need to use the same formulas:

\( x = r \cos \theta \), \( y = r \sin \theta \), and \( x^2 + y^2 = r^2 \).

Example 7: Write the equation using polar coordinates \((r, \theta)\).

#68) \( x^2 + y^2 = x \)

\[
\begin{align*}
   #70 & \quad y^2 = 2x \\
   #72 & \quad 4x^2 y = 1
\end{align*}
\]
Section 10.2 – Polar Equations and Graphs

In this section we will learn how to graph various polar equations in the polar coordinate system. The main "shapes" that we will be graphing are circles, lines, cardioids, limaçons, roses, and lemniscates.

**LINES**

- If you are given a polar equation such as $\theta = \alpha$, then you merely draw a line through the pole that makes an angle of $\alpha$ with the polar axis.
- If you have a polar equation of the form $r \cos \theta = a$, then recall that $x = r \cos \theta$, so in rectangular coordinates that would become $x = a$, which is the equation of a ________________ line.
- For a polar equation of the form $r \sin \theta = b$, recall that $y = r \sin \theta$, so that becomes the equation $y = b$, which is the equation of a ________________ line.

**Example 1:** Graph the equation.

a) $\theta = -\frac{\pi}{4}$

b) $r \sin \theta = -2$

c) $r \cos \theta = 4$

**CIRCLES**

- Circles of the form $r = a$, where $a$ is a positive number (notice there is no $\theta$ in the equation!), form circles centered at the pole with a radius of $a$. **Example (a):** Graph $r = 4$.
- Circles of the form $r = \pm 2a \cos \theta$ have their centers on the ________________ and have a radius of $a$. If the equation is positive, then the circle is on the right side of the pole. If the equation is negative, then the circle is on the left side of the pole. These circles pass through the pole. **Example (b):** Graph $r = -4 \cos \theta$.
- Circles of the form $r = \pm 2a \sin \theta$ have their centers on the ________________ and also have a radius of $a$. If the equation is positive, then the circle is ________________ the pole. If the equation is negative, then the circle is ________________ the pole. These circles pass through the pole. **Example (c):** Graph $r = 2 \sin \theta$.
CARDIOIDS

Cardioids, such as the one shown in the figure on the right, got their name because the shape resembles a ________________. Cardioids pass through the pole, and the distance from the pole to the farthest point on the major axis is equal to $2a$. Also, the graph will pass through $a$ and $-a$ on the minor axis. Next to the box below, label where the main portion of the cardioid will appear (above, below, to the left, or to the right of the pole).

Cardioids are characterized by equations of the form:

- $r = a(1 + \cos \theta)$
- $r = a(1 + \sin \theta)$
- $r = a(1 - \cos \theta)$
- $r = a(1 - \sin \theta)$

where $a > 0$. The graph of a cardioid passes through the pole.

Once you have established the general shape and orientation of a cardioid, you must plot points on the graph by choosing various values of $\theta$ and finding out what $r$ values they result in.

**Example 2:** Graph the cardioid $r = 2 - 2\cos \theta$.

First, we note that the cardioid will be centered on the ______________ (because of the $\cos \theta$) and the main portion will be to the ___________ of the pole (because of the negative sign). We also need to factor out the 2, resulting in the equation ______________________________. We can see that $a = 2$, and thus the farthest point on the main axis will be a distance of __________ away from the pole. At this point we know the general shape of the cardioid. Now we need to find specific points on the graph. The easiest way to do this is to change your graphing calculator MODE from Function to POLAR. Click on $Y=$ then insert the equation. Now go to TblSet and have the TblStart be 0 and the $\Delta$Tbl be $\pi/6$. Then click on TABLE to see the $r$-values that correspond to each value of $\theta$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r = 2 - 2\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\pi/6$</td>
<td></td>
</tr>
<tr>
<td>$\pi/3$</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td></td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td></td>
</tr>
<tr>
<td>$5\pi/6$</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
</tr>
</tbody>
</table>
LIMAÇONS

There are 2 types of limaçons: those with an inner loop and those without an inner loop.

Limaçons without an inner loop are characterized by equations of the form

\[
\begin{align*}
  r &= a + b \cos \theta \\
  r &= a + b \sin \theta \\
  r &= a - b \cos \theta \\
  r &= a - b \sin \theta
\end{align*}
\]

where \(a > 0, b > 0, \) and \(a > b\). The graph of a limaçon without an inner loop does not pass through the pole.

Notice in the box above the specification that \(a > b\) (the constant is bigger than the coefficient). This is precisely what makes this limaçon not have an inner loop. For limaçons that do have an inner loop, \(a < b\) (so the constant is smaller than the coefficient). For instance, in the equation \(r = 3 + 2 \cos \theta\), \(a = 3\) and \(b = 2\). Since \(a > b\), this limaçon does NOT have an inner loop. But for the equation \(r = 2 + 3 \cos \theta\), \(a = 2\) and \(b = 3\), so since \(a < b\), this limaçon DOES have an inner loop. What about the case where \(a = b\), such as \(r = 2 + 2 \cos \theta\)? We can rewrite the equation as _______________ and we realize that it is actually a ____________________.

Limaçons follow the same rules as cardioids in terms of where the main part (or bulk) of the graph will appear. For \(+ \cos \theta\) it will be to the __________ of the pole. For \(- \cos \theta\) it will be to the __________ of the pole. For \(+ \sin \theta\) it will be __________ the pole, and for \(- \sin \theta\) it will be __________ the pole.

Although the graphs of limaçons without an inner loop do NOT pass through the pole, the graphs of limaçons with an inner loop pass through the pole twice. In the graph on the right, pick any starting point and trace the limaçon to see how this happens.

Limaçons will always pass through \(a + b\) on the major axis and through \(a\) and \(-a\) on the minor axis. Additionally, limaçons without inner loops will pass through \(a - b\) on the other side of the pole from the bulk of the graph, and for those limaçons that do have inner loops, the lengths of the inner loops are \(b - a\).

Example 3: What are the equations of the limaçons shown at the top of the page?

Example 4: What is the equation of the limaçon with an inner loop shown above? \(r = \) ____________________
ROSE CURVES

Rose curves are characterized by equations of the form:

\[ r = a \cos(n\theta), \quad r = a \sin(n\theta), \quad a \neq 0 \]

and have graphs that are rose shaped.

If \( n \neq 0 \) is even, the rose has \( 2n \) petals;
if \( n \neq \pm 1 \) is odd, the rose has \( n \) petals.

Notice in the equations to the left that \( \theta \) has a coefficient. If \( \theta \)
does not have a coefficient (such as \( r = 2\cos\theta \), for instance),
the graph is not a rose curve. We have actually already learned
how to graph this kind of equation. What will the graph of
\( r = 2\cos\theta \) be? ____________________________

So we realize that in order to make a rose curve, \( \theta \) must have a coefficient, \( n \).

- If \( n \) is EVEN, then the rose will have \( 2n \) petals. For \( \cos(\text{even } \theta) \), petals will lie on both the \( x \) and \( y \)-axes. For \( \sin(\text{even } \theta) \), no petals will lie on either axis.
- If \( n \) is ODD, then the rose will \( n \) petals. For \( \cos(\text{odd } \theta) \), one petal will lie on the ______________ and for \( \sin(\text{odd } \theta) \), one petal will lie directly on the ____________.
- Lastly, the length of each petal will be equal to \( a \) (the coefficient).

Example 5: Give the equations of each of the rose curves shown below.

Example 6: Graph the polar equation \( r = 3\cos(4\theta) \).

First, we identify this as a ________________________.

Since \( n \) is ____________, we know there will be __________ petals. We see that one petal will lie
directly on ______________________ and the length of each petal will be ________.
LEMNISCATES

Lemniscates are propeller-shaped graphs, as shown in the figures on the right.

where $a \neq 0$, and have graphs that are propeller shaped.

Notice that the $r$ is SQUARED in these equations. In order to put them into your calculators, you need to take the square root of both sides and put the square root into your calculator. For instance, to get the graph of $r^2 = 6\sin(2\theta)$, I typed $r = \sqrt{6\sin(2\theta)}$ into my calculator. To graph the equation on the right, what would you put into your calculator? ________________

Notice that the equations containing sine create lemniscates in Quadrants 1 and 3, whereas the ones containing cosine create lemniscates along the x-axis. Also, the length of each loop is $a$, which is the square root of the _________________.

Example 4: Identify the type of graph and the key features of each graph. Then graph on polar graph paper.

a) $r = 2 + 4\cos \theta$  

b) $r = 2\sin(5\theta)$

c) $r = 3 - \sin \theta$  

d) $r^2 = 9\sin(2\theta)$

e) $r = 6\cos \theta$  

f) $r \cos \theta = -2$
Section 10.3 – The Complex Plane; De Moivre's Theorem

REVIEW OF COMPLEX NUMBERS FROM COLLEGE ALGEBRA

You learned about complex numbers of the form \( a + bi \) in your college algebra class. You should remember that "i" is the imaginary unit and it is defined as \( i = \sqrt{-1} \). By squaring both sides, we see that \( i^2 = \boxed{1} \). In a complex number \( a + bi \), \( a \) is called the ________ part and \( b \) is called the ________ part. You should also recall that \( a - bi \) is called the ______________________ of \( a + bi \). If we multiply a complex number by its conjugate, we get the following result: \( (a + bi)(a - bi) = a^2 + b^2 = a^2 - b^2(-1) = a^2 + b^2 \). For example, \( (2 - 5i)(2 + 5i) = \boxed{1} \).

In your college algebra class, you never learned how to graph complex numbers. We will do that now.

THE COMPLEX PLANE

We will now define a complex number as "\( z \)" , where \( z = x + yi \) replaces our typical \( a + bi \) notation. Therefore, \( x \) becomes the real part, which we will plot on the real axis, and \( y \) is the imaginary part, which we will plot on the imaginary axis. The distance from the origin to the point \( z \) is called the magnitude of \( z \). Using the distance formula \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \), we find that the magnitude \( |z| = \sqrt{(x - 0)^2 + (y - 0)^2} \) \( \rightarrow |z| = \sqrt{x^2 + y^2} \).

Since \( z = x + yi \), we will denote its conjugate as \( \overline{z} \) (called "z-bar"). The product of \( z \) and \( \overline{z} \) is:

\[
z \cdot \overline{z} = (x + yi)(x - yi) = x^2 + y^2 \rightarrow |z| = \sqrt{x^2 + y^2}.
\]

**Example 1:** If \( z = 3 - 2i \), Graph \( z \) and find \( \overline{z} \) and \( |z| \).

CONVERTING A COMPLEX NUMBER FROM RECTANGULAR FORM TO POLAR FORM

Recall that \( x = r \cos \theta \) and \( y = r \sin \theta \). Therefore, we can write a complex number as \( z = (r \cos \theta) + (r \sin \theta)i \), and if we factor out the \( r \), we can also write it as \( z = r (\cos \theta + i \sin \theta) \). Recall also that \( r^2 = x^2 + y^2 \), so \( r = \sqrt{x^2 + y^2} \). Now, since the magnitude of \( z \) is \( |z| = \sqrt{x^2 + y^2} \), that means that \( |z| = r \). Lastly, recall that \( \tan \theta = \frac{y}{x} \), so \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \).
Example 2: Plot each complex number in the complex plane and write it in polar form. Express the argument in degrees.

a) $-1 + i$

Start by plotting the point in the complex plane. Since $x = -1$ and $y = 1$, this point is in Quadrant ______.

Now we find $r$ and $\theta$: $r = \sqrt{(-1)^2 + (1)^2} \rightarrow r = \sqrt{2}$.

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) \rightarrow \theta = \tan^{-1}(-1) \rightarrow \theta = 135^\circ$.

Thus, the complex number $z = -1 + i$ is equivalent to $z = \sqrt{2}(\cos135^\circ + i\sin135^\circ)$ in polar form.

b) $2 + \sqrt{3}i$

Start by plotting the point in the complex plane. Since $x = ____$ and $y = ____$, this point is in Quadrant ______.

Now we find $r$ and $\theta$: $r = \sqrt{(2)^2 + (\sqrt{3})^2} \rightarrow r = ____$.

$\theta = \tan^{-1}\left(____\right) \rightarrow \theta = ____$.

Thus, the complex number $z = 2 + \sqrt{3}i$ is equivalent to $z = ____$ in polar form.

Example 3: Plot the point in the complex plane and write each complex number in rectangular form.

a) $2(\cos210^\circ + i\sin210^\circ)$

Recall that a complex point in polar coordinates has the form $z = r(\cos \theta + i\sin \theta)$, so for this problem, $r = 2$ and $\theta = 210^\circ$. This point is in Quadrant ______. Note that cosine and sine are both __________ in this quadrant and 210$^\circ$ has a reference angle of _______.

Thus, $\cos210^\circ = -\cos30^\circ = -____$ and $\sin210^\circ = -\sin30^\circ = -____$.

Substituting these values into the original point gives:

$$2(\cos210^\circ + i\sin210^\circ) = 2\left(-____ + i(-____)\right) \rightarrow$$
Example 3 (continued): Plot the point in the complex plane and write each complex number in rectangular form.

b) \(4 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)\) For this problem, \(r = \) _______ and \(\theta = \) _______. This is a __________________________ angle

so we need to use the ___________________________ to find the values of \(\cos \frac{\pi}{2} = \) _______ and

\(\sin \frac{\pi}{2} = \) __________. So \(z = 4(_______+i_______) = \) __________

FINDING PRODUCTS AND QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

We will use these formulas to find a product or quotient of complex numbers written in polar form. These formulas are derived by multiplying \(z_1\) by \(z_2\) and then applying the sum/difference formulas we learned back in section 8.4.

Let \(z_1 = r_1(\cos \theta_1 + i \sin \theta_1)\) and \(z_2 = r_2(\cos \theta_2 + i \sin \theta_2)\) be two complex numbers. Then

\[
z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \quad (5)
\]

If \(z_2 \neq 0\), then

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad (6)
\]

Example 4: Find \(zw\) and \(\frac{z}{w}\). Leave your answers in polar form.

a) \(z = \cos 120^\circ + i \sin 120^\circ, w = \cos 100^\circ + i \sin 100^\circ\)

For both \(z\) and \(w\), \(r = 1\). \(\theta_z = 120^\circ\) and \(\theta_w = 100^\circ\).

\[
zw = r_zr_w[\cos(\theta_z + \theta_w) + i \sin(\theta_z + \theta_w)]
\]

\[
z = \frac{r_z}{r_w}[\cos(\theta_z - \theta_w) + i \sin(\theta_z - \theta_w)]
\]

\[
z = \frac{1}{w}[\cos(120^\circ - 100^\circ) + i \sin(120^\circ - 100^\circ)]
\]

\[
z = \cos 20^\circ + i \sin 20^\circ
\]

b) \(z = 4 \left( \cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8} \right), w = 2 \left( \cos \frac{9\pi}{16} + i \sin \frac{9\pi}{16} \right)\)

\(r_z = \) _______, \(\theta_z = \) _______, \(r_w = \) _______, \(\theta_w = \) _______
USING DE MOIVRE’S THEOREM

De Moivre’s Theorem allows us to find powers of complex numbers using the formula on the right. It is a very simple formula to use. Let’s look at two examples.

Example 5: Write each expression in the standard form $a + bi$.

a) $\left(\sqrt[4]{2} \left(\cos \frac{5\pi}{16} + i \sin \frac{5\pi}{16}\right)\right)^4$. In this case, $n = 4$, $r = \sqrt{2}$, and $\theta = \frac{5\pi}{16}$.

So $z^4 = \left(\sqrt{2}\right)^4 \left[\cos \left(4 \cdot \frac{5\pi}{16}\right) + i \sin \left(4 \cdot \frac{5\pi}{16}\right)\right] = 4 \left[\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right] = 4 \left[-\frac{\sqrt{2}}{2} + i \left(-\frac{\sqrt{2}}{2}\right)\right] = -2\sqrt{2} - 2\sqrt{2} \cdot i$

b) $\left(\sqrt[6]{3} \left(\cos \frac{5\pi}{18} + i \sin \frac{5\pi}{18}\right)\right)^6$. $n = _______, r = _________, and \theta = ________.

So $z^6 =$

c) $(\sqrt{3} - i)^6$.

We first have to write this complex number in polar form. Since $x = \sqrt{3}$ and $y = -1$, this point is in Q_____.

Recall that $r = \sqrt{x^2 + y^2}$. So $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = ________.

$\theta = \tan^{-1} \left(\frac{y}{x}\right) = \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right)$. Since $\theta$ must be in Q4, then $\theta = ________.$

So $z = \sqrt{3} - i$ is equivalent to $z = _______{\cos \left(-\right) + i \sin \left(-\right)}$ in polar form.

Then $z^6 = r^6 \left[\cos(6\theta) + i \sin(6\theta)\right] = 2^6 \left[\cos \left(\theta \cdot \left(-\frac{\pi}{\theta}\right)\right) + i \sin \left(\theta \cdot \left(-\frac{\pi}{\theta}\right)\right)\right]$

$= 64 \left[\cos(-\pi) + i \sin(-\pi)\right] \cdot \cos \text{is even, sin is odd} \rightarrow 64 \left[\cos(\pi) - i \sin(\pi)\right] = 64 \left[1 - i(0)\right] = -64$
FINDING COMPLEX ROOTS

Here we learn how to find roots of complex numbers.

**Finding Complex Roots**

Let \( w = r(\cos \theta_0 + i \sin \theta_0) \) be a complex number and let \( n \geq 2 \) be an integer. If \( w \neq 0 \), there are \( n \) distinct complex roots of \( w \), given by the formula:

\[
z_k = \sqrt[n]{r} \left[ \cos \left( \frac{\theta_0}{n} + \frac{2k\pi}{n} \right) + i \sin \left( \frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right]
\]

where \( k = 0, 1, 2, \ldots, n - 1 \).

Note that if you are working in degrees, then you would use 360° instead of 2\( \pi \), so the formula becomes:

\[
z_k = r^{\frac{1}{n}} \left[ \cos \left( \frac{\theta_0}{n} + \frac{360^\circ k}{n} \right) + i \sin \left( \frac{\theta_0}{n} + \frac{360^\circ k}{n} \right) \right],
\]

where I chose to rewrite \( \sqrt[n]{r} \) as \( r^{\frac{1}{n}} \).

This is not the clearest formula in the world, so let's look at an example to see how it works.

**Example 6:** Find all the complex cube roots of \( -8 - 8i \). Leave your answers in polar form with the argument in degrees.

First we express the number in polar form using degrees. With \( x = -8 \) and \( y = -8 \), this number is in Q ______.

\[
r = \sqrt{(-8)^2 + (-8)^2} = \sqrt{64 + 64} = \sqrt{128} = 2^7 = 2^{\frac{7}{2}}
\]

\[
\theta = \tan^{-1}\left(\frac{-8}{-8}\right) = \tan^{-1}(1) \rightarrow \theta = 225^\circ.
\]

Because we are finding "cube" roots, use \( n = 3 \) in the formula:

\[
z_k = r^{\frac{1}{3}} \left[ \cos \left( \frac{\theta_0}{3} + \frac{360^\circ k}{3} \right) + i \sin \left( \frac{\theta_0}{3} + \frac{360^\circ k}{3} \right) \right]
\]

\[
z_k = \left(2^{\frac{7}{2}}\right)^{\frac{1}{3}} \left[ \cos \left( \frac{225^\circ}{3} + \frac{360^\circ k}{3} \right) + i \sin \left( \frac{225^\circ}{3} + \frac{360^\circ k}{3} \right) \right]
\]

\[
z_k = 2^{\frac{7}{6}} \left[ \cos \left( 75^\circ + 120^\circ k \right) + i \sin \left( 75^\circ + 120^\circ k \right) \right]
\]

Multiply the exponents together.

Now we evaluate \( z \) for \( k = 0, 1, \) and \( 2 \).

\[
k = 0: z_0 = 2^{\frac{7}{6}} \left[ \cos \left( 75^\circ + 120^\circ \cdot 0 \right) + i \sin \left( 75^\circ + 120^\circ \cdot 0 \right) \right] = 2^{\frac{7}{6}} \left[ \cos 75^\circ + i \sin 75^\circ \right]
\]

\[
k = 1: z_1 = 2^{\frac{7}{6}} \left[ \cos \left( 75^\circ + 120^\circ \cdot 1 \right) + i \sin \left( 75^\circ + 120^\circ \cdot 1 \right) \right] = 2^{\frac{7}{6}} \left[ \cos 195^\circ + i \sin 195^\circ \right]
\]

\[
k = 2: z_2 = 2^{\frac{7}{6}} \left[ \cos \left( 75^\circ + 120^\circ \cdot 2 \right) + i \sin \left( 75^\circ + 120^\circ \cdot 2 \right) \right] = 2^{\frac{7}{6}} \left[ \cos 315^\circ + i \sin 315^\circ \right]
\]
Section 10.4 – Vectors

A vector is represented by using a ray, or arrow, that starts at an initial point and ends at a terminal point. Your textbook will always use a bold letter to indicate a vector (such as \( \mathbf{v} \)), but since we can’t write in bold, we put an arrow over the letter that represents a vector (such as \( \vec{v} \) or \( \vec{\mathbf{v}} \)). When writing, I will use the latter notation (like the top half of a horizontal arrow) because it is the quickest to write and also because this is the notation that physicists typically use, and we learn about vectors mainly for use in future physics classes.

A vector has both a magnitude (length) and direction. The arrowhead is always at the terminal point of the vector and it shows you which direction a vector has traveled. Two vectors are equal if they have the same magnitude and direction, even if they don’t have the same initial and terminating points. All three vectors in the figure to the right are equal to one another.

Two vectors that have the same magnitude but exactly opposite directions are denoted as \( \vec{v} \) and \( -\vec{v} \). When we talk about the direction of a vector we are really talking about two things: the angle the vector makes (you might think of this as its slope) and also whether it is pointing up, down, left, or right. To have "opposite directions" the slope must be the same (so the vectors must be \( \text{_______________} \) but the pointing direction must be opposite. Let’s look at some examples.

In the top box of the figure to the left, both vectors are clearly the same length, so they have the same magnitude. Additionally, the vectors are parallel to one another, so they have the same slope. One is pointing right while the other is pointing left. So these are opposite vectors. In the second box, the vectors have the same magnitude and the same slope. One is pointing up and the other is pointing down. So these are opposite vectors also. In the last box, the vectors have the same magnitude, but they are not parallel. Because they do not have the same slope, these are not opposite vectors.

ADDITIONING VECTORS

To add vectors \( \vec{v} + \vec{w} \), you place the initial point of \( \vec{w} \) at the terminal point of \( \vec{v} \). Then the resultant sum vector \( \vec{v} + \vec{w} \) is the vector that takes you from the initial point of \( \vec{v} \) to the terminal point of \( \vec{w} \), as shown in the figure to the right. The order you add vectors in does not matter, so the vector \( \vec{v} + \vec{w} \) is the same as the vector \( \vec{w} + \vec{v} \). Try this in the figure to the right. Draw \( \vec{w} \), then add \( \vec{v} \) to it. Notice that the vector you get has the exact same magnitude and direction as the vector \( \vec{v} + \vec{w} \) does. This means that vector addition is commutative.
Vector addition is also associative, as shown in the figure to the right. Whether you add \( \vec{u} + \vec{v} \) then \( \vec{w} \), or \( \vec{v} + \vec{w} \) then \( \vec{u} \), or \( \vec{u} + \vec{w} \) then \( \vec{v} \) (not shown), the resultant vector will still have the same magnitude and direction.

The zero vector, \( \vec{0} \), has the property that \( \vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v} \). Also, if you add a vector, \( \vec{v} \), to its opposite vector, \( -\vec{v} \), you get the zero vector: \( \vec{v} + (-\vec{v}) = \vec{0} \).

The difference of two vectors is defined as follows: \( \vec{v} - \vec{w} = \vec{v} + (-\vec{w}) \). So we never actually subtract vectors; instead, we add the opposite of the second vector. \( \vec{v} \) and \( \vec{w} \) are shown in the figure to the left. To find the difference \( \vec{v} - \vec{w} \), we add \( (-\vec{w}) \) to the terminal point of \( \vec{v} \). The resultant vector \( \vec{v} - \vec{w} \) starts at the initial point of \( \vec{v} \) and ends at the terminal point of \( (-\vec{w}) \).

**MULTIPLYING VECTORS BY NUMBERS**

In the world of vectors, we call real numbers scalars. Scalars have only magnitude; they do not have direction. If we multiply a positive scalar \( \alpha \) times a vector \( \vec{v} \), the magnitude becomes \( \alpha \) times the magnitude of the vector, but the direction is unchanged. If we multiply a negative scalar \( -\alpha \) times a vector \( \vec{v} \), the magnitude becomes \( |\alpha| \) times the magnitude of the vector and the direction changes to the opposite of \( \vec{v} \). Draw the vector \( -2\vec{v} \) in the figure above.

**GRAPHING VECTORS**

Example 1: Given the vectors \( \vec{u} \), \( \vec{v} \), and \( \vec{w} \) shown in the left-hand box below, graph the following:

- a) \( \vec{v} + \vec{w} \)
- b) \( \vec{u} - \vec{v} \)
- c) \( 2\vec{w} \)
- d) \( -3\vec{v} \)
- e) \( \vec{u} - 2\vec{w} + \vec{v} \)

**MAGNITUDES OF VECTORS AND THE UNIT VECTOR**

As we've already learned, the magnitude of a vector is its length. We will use the notation \( \|\vec{v}\| \) to represent the magnitude of \( \vec{v} \). A magnitude is always positive (since there is no such thing as a negative length).

A vector \( \vec{u} \) that has a magnitude of 1 (\( \|\vec{u}\| = 1 \)) is called a unit vector.
FINDING A POSITION VECTOR

In order to graph the vectors in the previous example, we had to count how many units right or left and up or down a vector traveled. For instance, from its initial point to its terminal point, the vector \( \mathbf{w} \) moved right 3 and down 2. If we assumed that the initial point was at the origin, then the terminal point would be at \( \text{___________} \). But because this is a vector, we use the notation \( \mathbf{v} = \langle a, b \rangle \), where \( a \) and \( b \) are called the components of the vector. A vector written in this manner \( \mathbf{v} = \langle a, b \rangle \) is called an algebraic vector. The specific case where a vector has its initial point at the origin is called a position vector. Recall that a unit vector has a magnitude of \( \text{_______} \). If we say that \( \mathbf{i} \) is the unit vector that points directly along the \( x \)-axis and \( \mathbf{j} \) is the unit vector that points directly along the \( y \)-axis (as shown in the figure to the right), then we can rewrite \( \mathbf{v} = \langle a, b \rangle \) as \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} \), which indicates we move \( a \) units to the right or left and \( b \) units up or down.

Label the vectors \( \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \) from the previous page in their component form, either \( \langle a, b \rangle \) or \( a\mathbf{i} + b\mathbf{j} \).

\[
\mathbf{u} = \text{___________} \quad \mathbf{v} = \text{___________} \quad \mathbf{w} = \text{___________}
\]

A vector whose initial point is not at the origin can be rewritten algebraically to find an equivalent position vector by subtracting the components of the initial point from the components of the terminal point.

Example 2: The vector \( \mathbf{v} \) has initial point \( P \) and terminal point \( Q \). Write \( \mathbf{v} \) in the form \( a\mathbf{i} + b\mathbf{j} \) (i.e. find its position vector).

(a) \( P = (-3, 2); Q = (6, 5) \) 
\[
\mathbf{v} = \langle 6 - (-3), 5 - 2 \rangle = \langle 9, 3 \rangle \rightarrow \mathbf{v} = 9\mathbf{i} + 3\mathbf{j}
\]

(b) \( P = (-1, 4); Q = (6, 2) \) 

(c) \( P = (1, 1); Q = (2, 2) \)

The magnitude of a vector \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} \) can be found using the formula \( \| \mathbf{v} \| = \sqrt{a^2 + b^2} \).

Example 3: Find \( \| \mathbf{v} \| \).

(a) \( \mathbf{v} = -5\mathbf{i} + 12\mathbf{j} \) 
(b) \( \mathbf{v} = -\mathbf{i} - \mathbf{j} \)
ADDING AND SUBTRACTING VECTORS ALGEBRAICALLY

Addition and subtraction of vectors is very straightforward: you add or subtract the like components together ($i$'s with $i$'s, $j$'s with $j$'s). To multiply a scalar $\alpha$ by a vector, you multiply the scalar by each component of the vector (you can think of it as distributing).

Example 4: If $v = 3i - 5j$ and $w = -2i + 3j$, find the following:

a) $3v - 2w$

$$3(3i - 5j) - 2(-2i + 3j) \rightarrow 9i - 15j + 4i - 6j \rightarrow (9 + 4)i + (-15 - 6)j \rightarrow 13i - 21j \text{ or } (13, -21)$$

b) $4v + 3w$

c) $\|v + w\|$

$$\|(3i - 5j) + (-2i + 3j)\| \rightarrow \|(3 - 2)i + (-5 + 3)j\| \rightarrow \|1i - 2j\| \frac{a=1, b=-2}{a^2 + b^2} \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

d) $\|v\| + \|w\|$

FINDING A UNIT VECTOR

In some applications it will be helpful to be able to find a unit vector (remember a unit vector has a magnitude of 1) that has the same direction as a given vector. To do this, you simply divide a vector by its magnitude. So $\hat{u} = \frac{v}{\|v\|}$.

Example 5: Find the unit vector in the same direction as $\vec{v}$.

a) $\vec{v} = -5\vec{i} + 12\vec{j}$

$\|\vec{v}\| = \sqrt{(-5)^2 + (12)^2} = \sqrt{169} = 13$

Then $\hat{u} = \frac{-5}{13} \vec{i} + \frac{12}{13} \vec{j}$

b) $\vec{v} = 2\vec{i} - \vec{j}$
FINDING A VECTOR FROM ITS MAGNITUDE AND DIRECTION

In many applications, a vector is described by its magnitude (often a speed or force) and direction (an angle, \( \alpha \)) rather than its \( \vec{i} \) and \( \vec{j} \) components. In these cases it is necessary to determine the algebraic form of the vector using the formula:

\[
\vec{v} = \| \vec{v} \| (\cos \alpha \vec{i} + \sin \alpha \vec{j}) = \| \vec{v} \| \cos \alpha \vec{i} + \| \vec{v} \| \sin \alpha \vec{j}
\]

Example 6: Write the vector \( \vec{v} \) in the form \( a\vec{i} + b\vec{j} \), given its magnitude \( \| \vec{v} \| \) and the angle \( \alpha \) it makes with the positive x-axis.

a) \( \| \vec{v} \| = 8, \ \alpha = 45^\circ \)

Since \( \cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2} \), we have

\[
\vec{v} = 8 \left( \frac{\sqrt{2}}{2} \vec{i} + \frac{\sqrt{2}}{2} \vec{j} \right) \rightarrow \vec{v} = 4\sqrt{2} \vec{i} + 4\sqrt{2} \vec{j}
\]

b) \( \| \vec{v} \| = 3, \ \alpha = 240^\circ \)

ANALYZING OBJECTS IN STATIC EQUILIBRIUM

Forces can be represented by vectors, and when two forces act simultaneously on an object, their components add together to create a resultant force \( \vec{F}_1 + \vec{F}_2 \). An object is said to be in static equilibrium if the object is at rest and the sum of all forces acting on the object is zero.

Example 7: #68) A weight of 800 pounds is suspended from two cables. The left cable makes an angle of 35\(^\circ\) and the right cable makes an angle of 50\(^\circ\) with the beam. What are the tensions in the two cables? (Tension is the magnitude of the force.)

Assuming the weight is stationary, there are three forces creating static equilibrium: the force of each cable (\( \vec{F}_1 \) and \( \vec{F}_2 \)), and the force of the weight itself (\( \vec{F}_3 \)). We need to determine the algebraic components of \( \vec{F}_1 \) and \( \vec{F}_2 \). We already know the magnitude (force) of the weight itself: it has a force of 800 pounds, being pulled straight down (due to gravity), so its force can be written as \( \vec{F}_3 = -800\vec{j} \).

We need to find the angle that each vector makes with the positive x-axis so we can plug them into the formula \( \vec{F} = \| \vec{F} \| \cos \alpha \vec{i} + \| \vec{F} \| \sin \alpha \vec{j} \).

Assuming the point where the weight attaches to the cables is the origin, label the angles in the figure above.

\[\alpha_1 = \quad \alpha_2 = \quad\]
Example 7 (continued):
The problem asks us to find the tension (magnitude of the force) in each cable, so we are trying to find $\|\vec{F}_1\|$ and $\|\vec{F}_2\|$.

For ease of writing, let: $X = \|\vec{F}_1\|$ and $Y = \|\vec{F}_2\|$. We will use a calculator to find decimal approximations of cosine and sine. We will have to express the forces of the cables as vectors: $\vec{F} = \|\vec{F}\| \cos \alpha \, \hat{i} + \|\vec{F}\| \sin \alpha \, \hat{j}$.

$\vec{F}_1 = X \cos 145^\circ \hat{i} + X \sin 145^\circ \hat{j} \rightarrow \vec{F}_1 = -0.8192 \, \hat{i} + 0.5736 \, \hat{j}$

$\vec{F}_2 = Y \cos 50^\circ \hat{i} + Y \sin 50^\circ \hat{j} \rightarrow \vec{F}_2 = 0.6428 \, \hat{i} + 0.766 \, \hat{j}$

Because these three forces are in static equilibrium, their sum must equal zero. This means that the sum of each component must equal zero.

$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (-0.8192X + 0.6428Y) \hat{i} + (0.5736X + 0.766Y - 800) \hat{j}$

We start by finding a relationship between $\|\vec{F}_1\|$ and $\|\vec{F}_2\|$ (X and Y) using the horizontal $\hat{i}$ component:

$\hat{i}$ component: $-0.8192X + 0.6428Y = 0 \rightarrow 0.6428Y = 0.8192X \rightarrow Y = 1.2744X$

We plug this result in for $Y$ when working with the vertical $\hat{j}$ component:

$\hat{j}$ component: $0.5736X + 0.766Y - 800 = 0 \rightarrow 0.5736X + 0.766(1.2744X) - 800 = 0 \rightarrow 0.5736X + 0.9761904X = 800 \rightarrow 1.5497904X = 800 \rightarrow X = \frac{516.2 \, \text{lb}}{\|\vec{F}_1\|}$

Since $Y = 1.2744X$, then $Y = 1.2744(516.2) \rightarrow Y = \frac{657.8 \, \text{lb}}{\|\vec{F}_2\|}$

Example 8: Repeat the previous problem if the left-hand cable is attached to the beam at a $30^\circ$ angle, the right-hand cable is attached to the beam at a $55^\circ$ angle, and the weight being suspended is 950 pounds.
The product of two vectors, called the **dot product**, results in a scalar answer (not a vector as you might expect). For this reason, the dot product is also called the **scalar product**. The definition of the dot product is shown in the box to the right.

The dot product is commutative, so \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).

The distributive property also applies to the dot product, so \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \).

**Example:** Find the dot product \( \mathbf{v} \cdot \mathbf{w} \) if \( \mathbf{v} = \langle 2,2 \rangle \) and \( \mathbf{w} = \langle 1,2 \rangle \).

**FINDING THE ANGLE BETWEEN TWO VECTORS**

The angle between two vectors can be found by using the formula in the box to the right. If the angle \( \theta \) between two vectors is 0 or \( \pi \), then the vectors are parallel. If the angle between them is \( \frac{\pi}{2} \) (90°), then the vectors are **orthogonal**. "Orthogonal" is another word for **perpendicular**, which means that the vectors meet at a ___________ angle. Two vectors are orthogonal if their dot product is zero.

**Example 1:** Find the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \) and state if the vectors are parallel, orthogonal, or neither.

a) \( \mathbf{u} = \mathbf{i} + \mathbf{j} ; \quad \mathbf{v} = -\mathbf{i} + \mathbf{j} \)

First we find the dot product, \( \mathbf{u} \cdot \mathbf{v} : \)

These vectors are ____________________________.

b) \( \mathbf{u} = 3\mathbf{i} - 4\mathbf{j} ; \quad \mathbf{v} = 4\mathbf{i} - 3\mathbf{j} \)

First we find the dot product, \( \mathbf{u} \cdot \mathbf{v} : \)

Next we find the magnitudes of \( \mathbf{u} \) and \( \mathbf{v} : \)

\[
\|\mathbf{u}\| = \sqrt{(\_\_\_\_)^2 + (\_\_\_\_)^2} = \_\_\_\_
\]

\[
\|\mathbf{v}\| = \sqrt{(\_\_\_\_)^2 + (\_\_\_\_)^2} = \_\_\_\_
\]

Now plug this information into the formula: \( \cos \theta = \_\_\_\_ \rightarrow \theta = \cos^{-1} \_\_\_\_ \)

These vectors are ____________________________
DECOMPOSING A VECTOR INTO TWO ORTHOGONAL VECTORS

Sometimes when given two vectors, \( \vec{v} \) and \( \vec{w} \), it is necessary to decompose \( \vec{v} \) into two parts: one part, \( \vec{v}_1 \), that is parallel to \( \vec{w} \), and another part, \( \vec{v}_2 \), that is orthogonal to \( \vec{w} \). The parallel part, \( \vec{v}_1 \), is called the **projection of \( \vec{v} \) onto \( \vec{w} \)**.

The decomposition of \( \vec{v} \) into \( \vec{v}_1 \) and \( \vec{v}_2 \), where \( \vec{v}_1 \) is parallel to \( \vec{w} \) and \( \vec{v}_2 \) is perpendicular to \( \vec{w} \), is

\[
\vec{v}_1 = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w}, \quad \vec{v}_2 = \vec{v} - \vec{v}_1
\]

**Example 2:** Decompose \( \vec{v} \) into \( \vec{v}_1 \) which is parallel to \( \vec{w} \), and \( \vec{v}_2 \) which is orthogonal to \( \vec{w} \).

**a)** \( \vec{v} = -3\hat{i} + 2\hat{j}, \quad \vec{w} = 2\hat{i} + \hat{j} \)

First we find \( \vec{v}_1 \). \( \vec{v} \cdot \vec{w} = (\vec{v} \cdot \vec{w}) = (-3)(2) + (2)(1) = -6 + 2 = -4 \), and \( \|\vec{w}\| = \sqrt{(2)^2 + (1)^2} = \sqrt{5} \).

So \( \vec{v} \cdot \vec{w} = \frac{-4}{\|\vec{w}\|^2} = \frac{-4}{5} \). Then \( \vec{v}_1 = \frac{-4}{5}(2\hat{i} + \hat{j}) \rightarrow \vec{v}_1 = \frac{-8}{5}\hat{i} - \frac{4}{5}\hat{j} \).

Now \( \vec{v}_2 = \vec{v} - \vec{v}_1 \rightarrow \vec{v}_2 = (-3\hat{i} + 2\hat{j}) - \left( \frac{-8}{5}\hat{i} - \frac{4}{5}\hat{j} \right) \rightarrow \vec{v}_2 = \left( -3 + \frac{8}{5} \right)\hat{i} + \left( 2 + \frac{4}{5} \right)\hat{j} \rightarrow \vec{v}_2 = \frac{-7}{5}\hat{i} + \frac{14}{5}\hat{j} \)

**b)** \( \vec{v} = \hat{i} - 3\hat{j}, \quad \vec{w} = 4\hat{i} - \hat{j} \)
In addition to the dot product, there is another type of product of two vectors. This additional product, called the **cross product**, applies only to three-dimensional vectors (also called “vectors in space”). Unlike the result of the dot product, which results in a scalar (a single number), the cross product of two vectors results in a **vector** as the answer. For this reason, the cross product is sometimes called the “vector product”.

The cross product of \( \vec{v} \) cross \( \vec{w} \) (\( \vec{v} \times \vec{w} \)) is the determinant of the 3 x 3 matrix that is set up as follows:

\[
\vec{v} \times \vec{w} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2
\end{vmatrix}
\]

Note that the vertical bars around the matrix indicate that you must find the **determinant** of the matrix. This determinant can be evaluated using either the “basket weave method” (shown on this page) or the “minors” method that is explained in our textbook (and also on the next page of these notes).

**FINDING A DETERMINANT USING THE BASKET WEAVE METHOD**

To find a determinant using the basket weave method, you start out by writing the first two columns to the right of the original cross product matrix. Then you take the products of the three primary diagonals (going from left to right) and sum them up, then take the products of the three secondary diagonals (going from right to left) and sum those up. The answer to the determinant is the first sum MINUS the second sum.

If we distribute the subtraction sign and combine like terms, we get:

\[
S_1 - S_2 = \left( b_1c_2\hat{i} + c_1a_2\hat{j} + a_1b_2\hat{k} \right) - \left( b_1a_2\hat{k} + c_1b_2\hat{i} + a_1c_2\hat{j} \right)
\]

Then if we factor out the unit vectors, the final result is:

\[
\vec{v} \times \vec{w} = (b_1c_2 - c_1b_2)\hat{i} + (c_1a_2 - a_1c_2)\hat{j} + (a_1b_2 - b_1a_2)\hat{k}
\]

Finding a Determinant Using Minors

The first thing you have to understand for this method is how to find the determinant of a 2 x 2 matrix. Fortunately, this is a simple task — you just find the product of the primary diagonal and subtract the product of the secondary diagonal:

\[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix} = ad - bc.
\]

A “minor” is the determinant of the matrix that results from covering up a row and a column from a larger matrix. To find a determinant of a general 3 x 3 matrix using minors, follow these steps:

1) Multiply the term in Row 1 Column 1 \( (a_{11}) \) by the determinant of the 2 x 2 matrix that results from covering up the first row and first column.

2) Multiply -1 times the term in Row 1 Column 2 \( (a_{12}) \) by the determinant of the 2 x 2 matrix that results from covering up the first row and second column.

3) Multiply the term in Row 1 Column 3 \( (a_{13}) \) by the determinant of the 2 x 2 matrix that results from covering up the first row and third column.

4) Add up the results from Steps 1 to 3:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33} \\
\end{vmatrix} = a_{11} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33} \\
\end{vmatrix} - a_{12} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33} \\
\end{vmatrix} + a_{13} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32} \\
\end{vmatrix}
\]

As mentioned, this is the process for finding the determinant of any 3 X 3 matrix using minors.

But for the specific case of the cross product of two vectors \( \vec{v} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k} \) and \( \vec{w} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k} \), we know that:

\[
\vec{v} \times \vec{w} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{vmatrix}
\]

So using the method of minors gives:

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{vmatrix} = \hat{i} \begin{vmatrix}
  b_1 & c_1 \\
  b_2 & c_2 \\
\end{vmatrix} - \hat{j} \begin{vmatrix}
  a_1 & c_1 \\
  a_2 & c_2 \\
\end{vmatrix} + \hat{k} \begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
\end{vmatrix}.
\]

Then evaluating each of the minor determinants gives:

\[
(b_2 c_1 - b_1 c_2) \hat{i} - (a_1 c_2 - c_1 a_2) \hat{j} + (a_1 b_2 - b_1 a_2) \hat{k}.
\]

Compare this to the boxed equation on the previous page. You will see that if you distribute the minus sign on the middle term of this equation, it is the same as the boxed equation from page 1.
It is YOUR CHOICE whether you use the basket weave or minors method for evaluating the determinant. You do NOT need to learn both methods, so just pick whichever one makes the most sense to you.

Example 1: Find the determinant of each 2 x 2 matrix.

(a) \[
\begin{vmatrix}
-2 & 5 \\
2 & -3 \\
\end{vmatrix}
\]

(b) \[
\begin{vmatrix}
6 & 5 \\
-2 & -1 \\
\end{vmatrix}
\]

Example 2: Find the cross product, \( \vec{v} \times \vec{w} \), of the given vectors.

(a) \( \vec{v} = -\vec{i} + 3\vec{j} + 2\vec{k} \)
    \( \vec{w} = 3\vec{i} - 2\vec{j} - \vec{k} \)

(b) \( \vec{v} = 3\vec{i} + \vec{j} + 3\vec{k} \)
    \( \vec{w} = \vec{i} - \vec{k} \)

Example 3: Find \( \vec{v} \times \vec{v} \) if \( \vec{v} = -3\vec{i} + 3\vec{j} + 2\vec{k} \).
GEOMETRIC PROPERTIES OF CROSS PRODUCTS

Recall from section 10.5 that two vectors are said to be orthogonal if they intersect at a 90° angle. Additionally, we learned that two vectors are orthogonal if their dot product equals zero.

Now we learn a fascinating fact: the vector that results from finding the cross product of two vectors is orthogonal to both vectors.

In other words, the vector \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both vectors \( \mathbf{u} \) and \( \mathbf{v} \).

Proof: If \( \mathbf{u} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k} \) and \( \mathbf{v} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k} \), then we know from our cross product definition at the bottom of page 2 that \( \mathbf{u} \times \mathbf{v} = (b_1 c_2 - c_1 b_2) \mathbf{i} - (a_1 c_2 - c_1 a_2) \mathbf{j} + (a_1 b_2 - b_1 a_2) \mathbf{k} \).

Now we check to see if \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) \) equals zero:

\[
(a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k}) \cdot \left[(b_1 c_2 - c_1 b_2) \mathbf{i} - (a_1 c_2 - c_1 a_2) \mathbf{j} + (a_1 b_2 - b_1 a_2) \mathbf{k}\right] \\
= a_1 (b_1 c_2 - c_1 b_2) - b_1 (a_1 c_2 - c_1 a_2) + c_1 (a_1 b_2 - b_1 a_2) \\
= a_1 b_1 c_2 - a_1 b_2 c_1 - a_1 b_2 c_1 + a_1 b_2 c_1 + a_1 b_2 c_1 - a_1 b_2 c_1 \\
= 0
\]

Since \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) \) equals zero, we know that \( \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \) are orthogonal vectors.

Similarly, \( \mathbf{v} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k} \) dotted with \( \mathbf{u} \times \mathbf{v} = (b_1 c_2 - c_1 b_2) \mathbf{i} - (a_1 c_2 - c_1 a_2) \mathbf{j} + (a_1 b_2 - b_1 a_2) \mathbf{k} \) equals zero:

\[
(a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k}) \cdot \left[(b_1 c_2 - c_1 b_2) \mathbf{i} - (a_1 c_2 - c_1 a_2) \mathbf{j} + (a_1 b_2 - b_1 a_2) \mathbf{k}\right] \\
= a_2 (b_1 c_2 - c_1 b_2) - b_2 (a_1 c_2 - c_1 a_2) + c_2 (a_1 b_2 - b_1 a_2) \\
= a_2 b_1 c_2 - a_2 b_2 c_1 - a_2 b_2 c_1 + a_2 b_2 c_1 + a_2 b_2 c_1 - a_2 b_2 c_1 \\
= 0
\]

Since \( \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) \) equals zero, we know that \( \mathbf{v} \) and \( \mathbf{u} \times \mathbf{v} \) are orthogonal vectors.

This orthogonal relationship is expressed in Theorem (8) in the box below. Additional geometric properties of the cross product are given as well.

\[
\begin{align*}
\text{Geometric Properties of the Cross Product} \\
\text{Let } \mathbf{u} \text{ and } \mathbf{v} \text{ be vectors in space.} \\
\mathbf{u} \times \mathbf{v} \text{ is orthogonal to both } \mathbf{u} \text{ and } \mathbf{v}. & \quad (8) \\
|\mathbf{u} \times \mathbf{v}| = ||\mathbf{u}|| ||\mathbf{v}|| \sin \theta, \text{ where } \theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v}. & \quad (9) \\
|\mathbf{u} \times \mathbf{v}| \text{ is the area of the parallelogram} & \quad (10) \\
\text{having } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \text{ as adjacent sides.} & \\
\mathbf{u} \times \mathbf{v} = \mathbf{0} \text{ if and only if } \mathbf{u} \text{ and } \mathbf{v} \text{ are parallel.} & \quad (11)
\end{align*}
\]
Let’s take a closer look at Theorem (10). It states that the area of a parallelogram created by adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is equal to the magnitude of the cross product of $\mathbf{u}$ and $\mathbf{v}$. (Remember that double vertical bars around a vector indicates that you are to find the magnitude of the vector, which is the square root of the sum of the squares of the components.)

Recall from geometry that the area of a parallelogram is equal to its base times its height (where the height is the distance from the base to the top, perpendicular to the base). Examining the parallelogram shown to the right, we can see that the height $(h)$ creates a right triangle within the parallelogram. The relationship between $h$, $\theta$, and $\|\mathbf{v}\|$, is $\sin(\theta) = \frac{h}{\|\mathbf{v}\|}$.

Therefore, $h = \|\mathbf{v}\|\sin(\theta)$, and thus the area of the parallelogram is $A = bh = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta)$, which, according to Theorem (9), is equal to the magnitude of $\mathbf{u}$ cross $\mathbf{v}$. Therefore, $A = bh = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta) = \|\mathbf{u} \times \mathbf{v}\|$.

**Example 4:** Find a vector orthogonal to both $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Check your answer.

**Example 5:** Find the area of the parallelogram with one corner at $P_1$ and adjacent sides $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, where $P_1 = (-2, 0, 2)$, $P_2 = (2, 1, -1)$, and $P_3 = (2, -1, 2)$. 